Class_05 Sep 22 More Polynomials

Monday, September 19, 2022 5:49 PM

Tonight's Class:

- **Chapter 1 Test Return**
- **Warm-up, 3.1, using small white-boards**
- **3.2 Polynomial Division**
- **3.3 Factor Theorem**
- **3.4 Polynomial Graphs**

What's one thing you want to remember from this chapter?

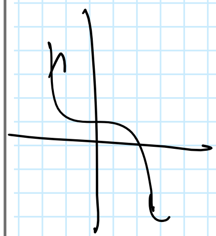
Could be one of these, or something else.

Domain/range of graph
Function or not?
Creating mapping notation
Listing changes and finding image point
Graphing something with several transformations
Graphing the inverse of graph
Finding the equation of the inverse

Warm-up, 3.1 - use small white-board

$f(x) = -x^5 + 6x^2 - 11x + 3$

1. Degree = 5



2. Leading coefficient = -1

3. End behavior = ↑ Q2 ↓ Q4

4. Possible number of x-intercepts = 1 up to 5

5. value of y-intercept = $(0, 3)$

6. domain = $\{x \mid x \in \mathbb{R}\}$

7. range = $\{y \mid y \in \mathbb{R}\}$

$f(x) = x^4 - 4x^3 + 2x^2 + x + 4$

1. Degree = 4

2. Leading coefficient = $+1$

3. End behavior = ↑ Q2 ↑ Q1

4. Possible number of x-intercepts = 0 up to 4

5. value of y-intercept = $(0, 4)$

6. domain = $\{x \mid x \in \mathbb{R}\}$

7. range = must graph it to know.
$\{y \mid y \geq -4.49, y \in \mathbb{R}\}$

3.2 The Remainder Theorem

Long division

divisor

quotient

$$\begin{array}{r} 42 \\ 6\overline{)253} \\ -24 \\ \hline 13 \\ -12 \\ \hline 1 \end{array}$$

dividend

remainder

Check: (divisor)(quotient) + remainder = dividend

$(6)(42) + 1 = 253$

Ways to write the result: $253 \div 6 = 42$, remainder 1 $\frac{253}{6} = 42$, remainder 1 $\frac{253}{6} = 42 + \left(\frac{1}{6}\right)$	**Check:** (Divisor)(Quotient) + Remainder = Dividend (6)(42) + 1 = 253

Using long division to divide a polynomial by a binomial

Divide $x^3 - 12x^2 - 42$ by $x - 3$

dividend   divisor

1) if dividend isn't in descending order, re-organize it!

2) if any degree isn't in the dividend, include it by putting it in with a 0 coefficient

quotient

$$\begin{array}{r} x^2 - 9x - 27 \\ x-3\overline{)x^3 - 12x^2 + 0x - 42} \\ -(x^3 - 3x^2) \qquad\qquad \\ \hline -9x^2 + 0x \qquad \\ -(-9x^2 + 27x) \qquad \\ \hline -27x - 42 \\ -(-27x + 81) \\ \hline -123 \end{array}$$

remainder

polynomial, the dividend   quotient   remainder

Express the result in the form $\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$ divisor

What restrictions are there on the variable?

$x \neq 3$ (that's so we don't try to divide by 0)

Verify (check) your answer.

$$\frac{x^3 - 12x^2 - 42}{x-3} = x^2 - 9x - 27 + \frac{-123}{x-3}$$

(Divisor)(Quotient) + Remainder = Dividend

$(x-3)(x^2 - 9x - 27) + -123 \stackrel{?}{=} x^3 - 12x^2 - 42$

$x^3 - 9x^2 - 27x - 3x^2 + 27x + 81 + -123$

$x^3 - 12x^2 - 42$ ✓

Division Statement
The result of dividing a polynomial $P(x)$ by a binomial of the form $x - a$ is:

$$\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}, \text{ where } \quad Q(x) \text{ is the quotient and } R \text{ is the remainder.}$$

Check: $P(x) = (x-a)Q(x) + R$
original polynomial = (divisor)(quotient) + remainder

there's no "x" term. Write in a "0x"

1a) Divide the polynomial $5x^3 + 3x^2 - 12$ by $x+2$ using ***long division.*** Express the result in the form $\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$

$$\begin{array}{r} 5x^2 - 7x + 14 \\ x+2 \overline{)\, 5x^3 + 3x^2 + 0x - 12} \\ -(5x^3 + 10x^2) \\ \hline -7x^2 + 0x \\ -(-7x^2 - 14x) \\ \hline 14x - 12 \\ -(14x + 28) \\ \hline -40 \end{array}$$

$$\frac{5x^3+3x^2-12}{x+2} = 5x^2 - 7x + 14 + \frac{-40}{x+2}$$

b) What restrictions are there on the variable?

$x \neq -2$ (comes from the divisor!)

c) Write the statement that can be used to check the division.

$5x^3 + 3x - 12 = (x+2)(5x^2 - 7x + 14) + -40$

d) Verify your answer.

$(x+2)(5x^2 - 7x + 14) + -40$

$= 5x^3 - 7x^2 + 14x + 10x^2 - 14x + 28 + -40$

$= 5x^3 + 3x^2 - 12$ ✓

2. Divide the polynomial $5x^3 + 3x^2 - 12$ by $x+2$ using ***synthetic division.*** Express the result in the form $\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$

(x+2)

$$\begin{array}{r|rrrr} -2 & 5 & 3 & 0 & -12 \\ - & & 10 & -14 & 28 \\ \hline \times \text{ (mult)} & 5 & -7 & 14 & -40 \end{array}$$

remainder

$$\boxed{5x^2 - 7x + 14 + \frac{-40}{x+2}}$$

Divide, using synthetic division:
$(2x^3+3x^2-5x+2)\div(x+3)$

$$\begin{array}{c|cccc} 3 & 2 & 3 & -5 & 2 \\ - & & 6 & -9 & 12 \\ \hline \times & 2 & -3 & 4 & -10 \end{array}$$

$2x^2-3x+4$ remainder $=-10$

$(x+3)(2x^2-3x+4)+-10$
$= 2x^3-3x^2+4x+6x^2-9x+12+-10$
$= 2x^3+3x^2-5x+2$ ✓

Find the value of $P(-3)$, for $P(x)=2x^3+3x^2-5x+2$.

$x=-3$

$P(-3)=2(-3)^3+3(-3)^2-5(-3)+2$
$=2(-27)+3(9)+15+2$
$=-10$

> ***Remainder Theorem***:
> When a polynomial, *P(x)*, is divided by a binomial, *x – a,* the remainder is $P(a)$.
>
> If $P(a)=0$, then the binomial $x-a$ ***is*** a factor of *P(x)*.
> If $P(a)\neq 0$, then the binomial $x-a$ is ***not*** a factor of *P(x)*.

Example
a) Use the Remainder Theorem to find the remainder when $P(x)=8x^3+4x^2-19$ is divided by $x+2$

How do we do it?

evaluate $P(-2)=8(-2)^3+4(-2)^2-19$
$=8(-8)+4(4)-19$
$=-64+16-19$
$=-48-19$
$=-67$

(this tells us $x+2$ is NOT a factor of P(x))

b) Check your answer by using synthetic division.

$$\begin{array}{c|cccc} 2 & 8 & 4 & 0 & -19 \\ - & & 16 & -24 & 48 \\ \hline \times & 8 & -12 & 24 & -67 \end{array}$$

c) Use the Remainder Theorem to find the remainder when $P(x)=8x^3+4x^2-19$ is divided by $x-1$.

evaluate $P(1)=8(1)^3+4(1)^2-19$
$=8+4-19$
$=12-19$
$=-7$

Table of P(x)

x	y
1	-7

TB, p 124

6. Use the remainder theorem to determine the remainder when each polynomial is

7. Determine the remainder resulting from

6. Use the remainder theorem to determine the remainder when each polynomial is divided by $x + 2$.

a) $x^3 + 3x^2 - 5x + 2$

b) $2x^4 - 2x^3 + 5x$

c) $x^4 + x^3 - 5x^2 + 2x - 7$

d) $8x^3 + 4x^2 - 19$

e) $3x^3 - 12x - 2$

f) $2x^3 + 3x^2 - 5x + 2$

7. Determine the remainder resulting from each division.

a) $(x^3 + 2x^2 - 3x + 9) \div (x + 3)$

b) $\dfrac{2t - 4t^3 - 3t^2}{t - 2}$

c) $(x^3 + 2x^2 - 3x + 5) \div (x - 3)$

d) $\dfrac{n^4 - 3n^2 - 5n + 2}{n - 2}$

p124

8. For each dividend, determine the value of k if the $\boxed{\text{remainder is 3.}}$

	Answers
a) $(x^3 + 4x^2 - x + k) \div (x - 1)$	$k = -1$
b) $(x^3 + x^2 + kx - 15) \div (x - 2)$	$k = 3$
c) $(x^3 + kx^2 + x + 5) \div (x + 2)$	$k = 2$
d) $(kx^3 + 3x + 1) \div (x + 2)$	$k = -1$

8b) $\overbrace{x^3 + x^2 + kx - 15}^{P(x)} \div x - 2$ gives a remainder of 3

We know that $P(2) = 3$, from the remainder theorem.

$$P(2) = 2^3 + 2^2 + k(2) - 15 = 3$$

$$8 + 4 + 2k - 15 = 3$$

$$12 + 2k - 15 = 3$$

$$2k - 3 = 3$$

$$+3 \qquad +3$$

$$\frac{2k}{2} = \frac{6}{2}$$

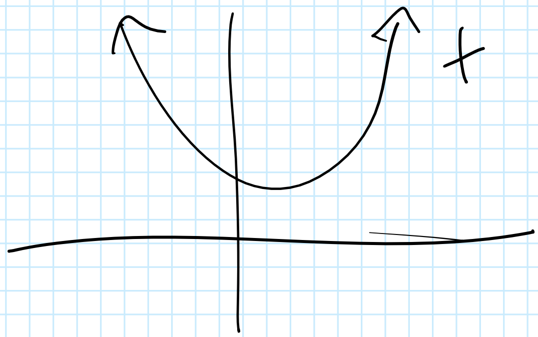
$$k = 3$$

So far we've learned:

- **How to tell what type of degree/leading coefficient a polynomial has, from its graph**
- **How to divide a polynomial by a binomial using long division or synthetic division**
- **How to find the remainder WITHOUT actually doing the division:**
  - **If we divide by x -a, the remainder is equal to P(a)**

even

even



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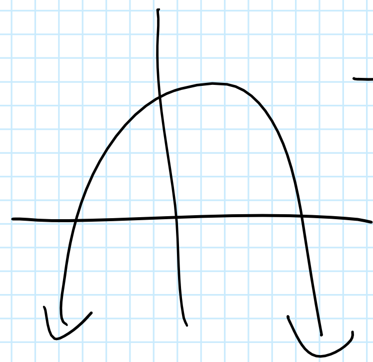

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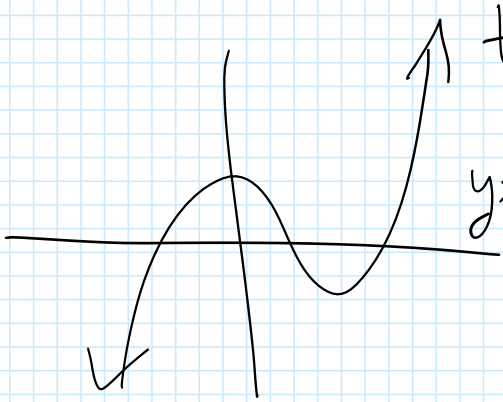
odd



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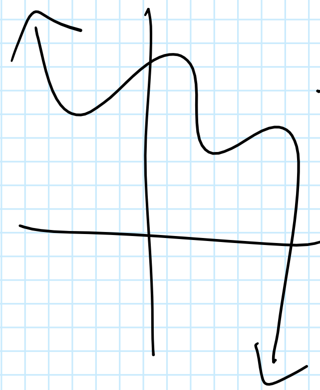
$y=x^3$



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TB, p 126

3.3

The Factor Theorem

Port of Vancouver

Focus on...

- factoring polynomials
- explaining the relationship between the linear factors of a polynomial expression and the zeros of the corresponding function
- modelling and solving problems involving polynomial functions

Each year, more than 1 million intermodal containers pass through the Port of Vancouver. The total volume of these containers is over 2 million twenty-foot equivalent units (TEU). Suppose the volume, in cubic feet, of a 1-TEU container can be approximated by the polynomial function $V(x) = x^3 + 7x^2 - 28x + 20$, where x is a positive real number. What dimensions, in terms of x, could the container have?

Did You Know?

An intermodal container is a standard-sized metal box that can be easily transferred between different modes of transportation, such as ships, trains, and trucks. A TEU represents the volume of a 20-ft intermodal container. Although container heights vary, the equivalent of 1 TEU is accepted as 1360 ft³.

If our goal is to factor a POLYNOMIAL completely, it helps us to know which BINOMIALS might possibly be factors.

Factor completely: $x^3 - 4x^2 + x + 6$

$= (\quad)(\quad)(\quad)$

These constants need to produce the "6" when multiplied together.

3.3 The Factor Theorem

For each binomial below, find the remainder when $P(x) = x^3 - 4x^2 + x + 6$ is divided by the binomial. Which of the following binomials are factors of $P(x) = x^3 - 4x^2 + x + 6$?

only possible things to try: $\pm 1, 6$ $2, 3$

a) $x+1$

Use remainder theorem:

$P(-1) = (-1)^3 - 4(-1)^2 + -1 + 6$
$= -1 - 4(1) + -1 + 6$
$= -1 - 4 - 1 + 6 = 0$

Yes, $x+1$ is a factor!

b) $x+2$

$P(-2) = (-2)^3 - 4(-2)^2 + -2 + 6$
$= -20$
not a factor

c) $x+3$

d) $x-1$

e) $x-2$

f) $x-3$

Factor Theorem: $x - a$ is a factor of a polynomial, $P(x)$, if and only if $P(a) = 0$

Example

Consider the polynomial: $P(x) = x^3 + 7x^2 - 28x + 20$.

a) Given that $P(-10) = 0$, what binomial must be a factor of $P(x)$?

$x + 10$ is a factor

$\pm 1, 20$ $2, 10$ $4, 5$ only possible things to try.

b) Factor $P(x)$ completely.

Use either synthetic or long division to divide:

$$\begin{array}{r l}
 & x^2 - 3x + 2 \\
x+10\ \big) & x^3 + 7x^2 - 28x + 20 \\
 & -(x^3 + 10x^2) \\
 & -3x^2 - 28x \\
 & -(-3x^2 - 30x) \\
 & 2x + 20 \\
 & -(2x + 20) \\
 & 0
\end{array}$$

OR

$$\begin{array}{c|cccc}
x+10 \quad 10 & 1 & 7 & -28 & 20 \\
- & \downarrow & 10 & -30 & 20 \\
\hline
\times & 1 & -3 & 2 & 0
\end{array}$$

$x^2 - 3x + 2$

$P(x) = x^3 - 7x^2 - 28x + 20$
$= (x+10)(x^2 - 3x + 2)$
$= (x+10)(x-2)(x-1)$

Integral Zero Theorem:

If $x - a$ is a factor of a polynomial with integral coefficients, $P(x)$, then a must divide evenly into the ***constant term*** of the polynomial $P(x)$.

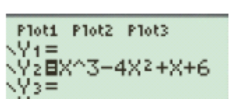
no decimals or fractions

only numbers that are factors of the constant can possibly give us factors of the polynomial

You can use a calculator's graphing table to quickly find factors of a polynomial:

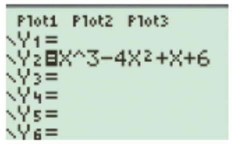
You need to be able to show how you can calculate the remainder, using substitution into P(x). However, it's good to know this, too:

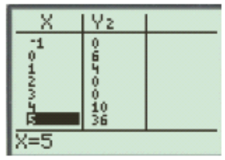
If you look at the table of values for the polynomial, it shows you what P(x) is for specific x values



If you look at the table of values for the polynomial, it shows you what P(x) is for specific x-values.

For example, below we can see that
P(-1) = 0 (so x +1 is a factor)
P(1) = 4 (so x - 1 is NOT a factor)
P(2) = 0 (so x - 2 is a factor)
P(3) = 0 (so x - 3 is a factor)
P(4) = 10 (so x - 4 is NOT a factor)
And so on....





x	x^3-4x^2+x+6
−2	−20
−1	0
0	6
1	4
2	0
3	0
4	10
5	36

Example

Factor fully without using technology: $2x^3-5x^2-4x+3$

a) According to the integral zero theorem, which values could possibly give factors of this polynomial?

$1, -1, 3, -3$ (only the numbers that are factors of the constant, 3)

b) Use the remainder and factor theorems to find a factor.

$$P(-1) = 2(-1)^3 - 5(-1)^2 - 4(-1) + 3$$
$$= -2 - 5(1) + 4 + 3$$
$$= -2 - 5 + 7$$
$$= 0$$

$x+1$

c) Use either long division or synthetic division to divide $2x^3-5x^2-4x+3$ by the factor found in part (b).

$x+1 \,\overline{)\, 2x^3-5x^2-4x+3}$ quotient $2x^2-7x+3$
$-(2x^3+2x^2)$
$-7x^2-4x$
$-(-7x^2-7x)$
$3x+3$
$-(3x+3)$
0

$x+1$

-1	2	-5	-4	3
	↓	2	-7	3
	2	-7	3	0

$2x^2-7x+3$

$P(x) = (x+1)(2x^2-7x+3)$
A B C

Decomp AC = 2(3)
= 6

mult = 6
add to B = -7 } $-6, -1$

$2x^2-6x-1x+3$
$= 2x(x-3) - 1(x-3)$
$(x-3)(2x-1)$

d) What is the fully factored form of $2x^3-5x^2-4x+3$?

$P(x) = (x+1)(x-3)(2x-1)$

e) Consider the equation $2x^3-5x^2-4x+3=0$, what are the solutions to this equation?

will do next class

f) Graph $2x^3-5x^2-4x+3$ on a graphing calculator. Find the values of its x-intercepts.

f) Graph $2x^3 - 5x^2 - 4x + 3$ on a graphing calculator. Find the values of its x-intercepts.

do next class

To factor fully:

1) **Look at the constant to see which numbers are worth trying. Only numbers that divide evenly into the constant term are possibilities.**
2) **When we get P(a) = 0, for some value, that means x - a is a factor.**
3) **Do the division, either by long division or synthetic division:**

$$\frac{P(x)}{x-a} = Q(x)$$

4) **The quotient you get will be of smaller degree than the original polynomial. Often, the original polynomial is degree 3, so the quotient will be of degree 2 and you can (hopefully) easily factor that. This means you would have something like this:**
P(x) = (x - a) Q(x)
= (x - a) ( )( )

Textbook, page 131 - try this one!

Your Turn

What is the fully factored form of $x^4 - 3x^3 - 7x^2 + 15x + 18$?

to try: $\pm 1, 18, 2, 9, 3, 6$

$$\begin{aligned} P(1) &= 1^4 - 3(1)^3 - 7(1)^2 + 15(1) + 18 \\ &= 1 - 3 - 7 + 15 + 18 \\ &= 24 \end{aligned}$$

$$\left.\begin{aligned} P(-1) &= (-1)^4 - 3(-1)^3 - 7(-1)^2 + 15(-1) + 18 \\ &= 1 - 3(-1) - 7(1) + -15 + 18 \\ &= 1 + 3 - 7 - 15 + 18 \\ &= 0 \checkmark \end{aligned}\right\} \Rightarrow x+1 \text{ is a factor}$$

$$(x+1) \quad \begin{array}{r|rrrrr} -1 & 1 & -3 & -7 & 15 & 18 \\ & \downarrow & -1 & 4 & 3 & -18 \\ \hline & 1 & -4 & -3 & 18 & 0 \end{array} \text{ — (Remainder)}$$

$$\underbrace{1 \quad -4 \quad -3 \quad 18}_{x^3 - 4x^2 - 3x + 18}$$

This is a cubic. We need to factor it. Notice the constant is still 18, and we already know $P(1) \neq 0$. (If it wasn't a factor of the quartic, it can't be of the cubic, either.)

$P(2) = 2^3 - 4(2)^2 - 3(2) + 18$ * I'm plugging into the cubic now, because it's a bit easier to work with

$= 8 - 4(4) - 6 + 18$

$= 8 - 16 - 6 + 18$

$= 4$

$P(-2) = (-2)^3 - 4(-2)^2 - 3(-2) + 18$

$$\left.\begin{aligned} &= -8 - 4(4) + 6 + 18 \\ &= -8 - 16 + 6 + 18 \\ &= 0 \end{aligned}\right\} \Rightarrow x + 2 \text{ is a factor.}$$

If you have a graphing calculator available, remember this can help:

- **Graph the polynomial**
- **Look at the table of values. Anywhere y = 0 there is an x-intercept.**
- **Each x-intercept will tell us a factor of the polynomial.**

x	y
-2	0
-1	0
3	0

---

x+2	2	1	−4	−3	18
	−	↓	2	−12	18
	×	1	−6	9	0

$\underbrace{1 \quad -6 \quad 9}_{x^2-6x+9}$

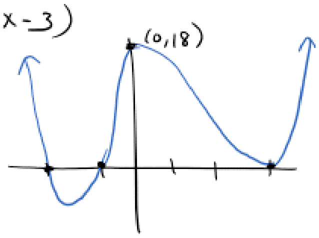
So, our original polynomial has factored into:

$(x+1)(x+2)(x^2-6x+9)$

let's factor this, too.

$= (x+1)(x+2)(x-3)(x-3)$

OR $\quad (x+1)(x+2)(x-3)^2$



If we graph, we have:

x-intercepts $\quad x=-1$

$x=-2$

$x=3$ (repeated)

y-int. $= (0,18)$

$x^4 - 3x^2 - 7x^2 + 15x + 18$ $\quad$ EVEN, ↑Q2 ↑Q1

For next class

Hand-in Assignment- work on the Chapter 3 Hand-in, try to do #1-7 and maybe even #8.

More Practice

- **TB (3.1) p 114: 1-3, 4ace, 6, 7, 9**
- **TB (3.2) p 124: 1-2, 3a, 4c, 5b, 6-8**
- **TB (3.3) p 133: 1-4, 5ace, 7bd, 9, 11**

http://www.mathsisfun.com/algebra/polynomials-division-long.html

For two more examples on how to do polynomial long division, watch part of this video:
https://www.youtube.com/watch?v=l6_ghhd7kwQ
- the first 3:42 of the video
- the example from 6:15-9:54 of the video
(The example in the middle of the video is a type of question we will not do in our class.)

Start preparing for Unit 1 Test (Chapters 1 and 3) on Thursday, September 29

Test format:

Around 40 marks possible

About 10 multiple-choice questions, the rest written questions.