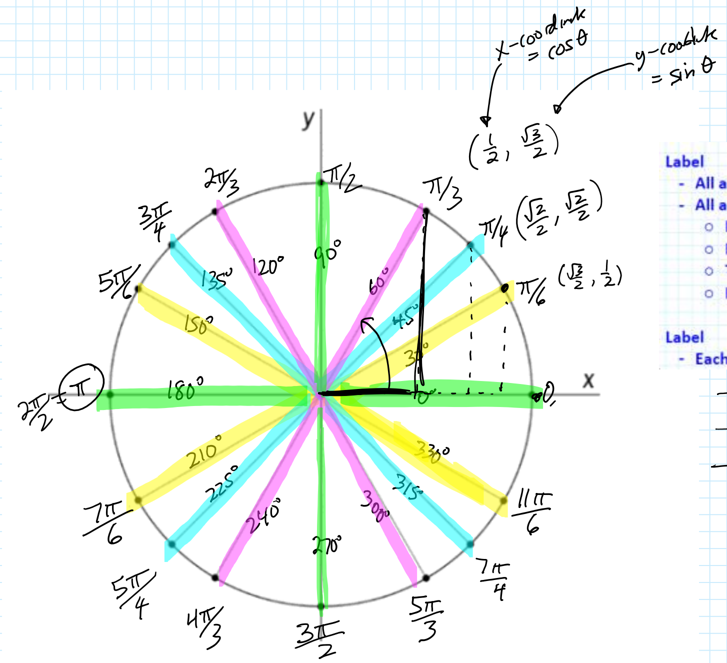
Class_08 May 16 Trig Ratios and Equations

Monday, May 15, 2023 2:59 PM

Tonight's Class:

- **Grab a white board, pen and eraser**
- **4.3 Trig Ratios**
- **4.4 Trigonometric Equations**



Label
- All angles, in degree measure
- All angles in radians,
  - First, ones with denominator "6"
  - Next, ones with denominator "4"
  - Then, ones with denominator "3"
  - Finally, ones with denominator "2"

Label
- Each point in Quadrant 1 with its x- and y-coordinates

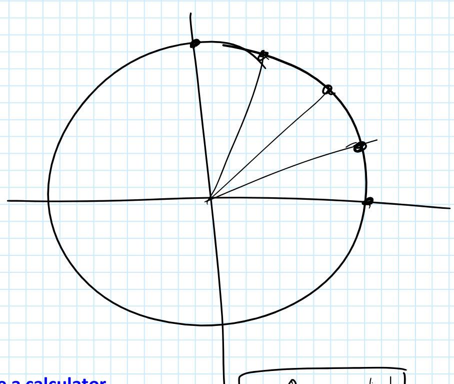
short = 1/2

medium = $\frac{\sqrt{2}}{2}$

long = $\frac{\sqrt{3}}{2}$

$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

A Trick to Remember Values on The Unit Circle - video

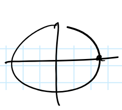


Quick Check-in - individual whiteboard

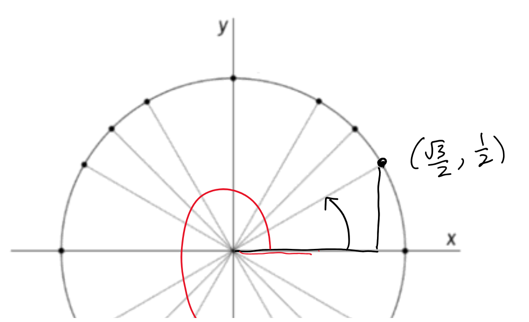
WB -
Use your unit circle to find these values exactly. Don't use a calculator.

$\boxed{\sin\theta = y\text{-coordinate}}$

1. $\sin 30^\circ = \frac{1}{2}$	2. $\sin 225^\circ = -\frac{\sqrt{2}}{2}$	3. $\sin 240^\circ = -\frac{\sqrt{3}}{2}$
4. $\sin 45^\circ = \frac{\sqrt{2}}{2}$	5. $\sin 150^\circ = \frac{1}{2}$	6. $\sin 300^\circ = -\frac{\sqrt{3}}{2}$
7. $\sin 0^\circ = 0$	8. $\sin 390^\circ = \frac{1}{2}$	9. $\sin 270^\circ = -1$
10. $\sin 210^\circ = -\frac{1}{2}$	11. $\sin 180^\circ = 0$	12. $\sin 330^\circ = -\frac{1}{2}$



1. $\cos 0 = 1$	2. $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$	3. $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$



$\sin 30^\circ = \frac{1}{2}$

1. $\cos 0 = 1$

2. $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

3. $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

4. $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

5. $\cos\left(\frac{3\pi}{2}\right) = 0$

6. $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

7. $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

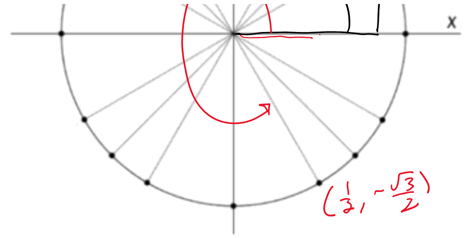
8. $\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$

9. $\cos\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2}$

10. $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

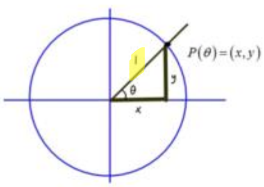
11. $\cos(\pi) = -1$

12. $\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}$



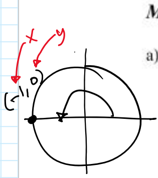
4.3 Trigonometric Ratios

We can use the unit circle diagram to get definitions for all six of the trigonometric ratios.



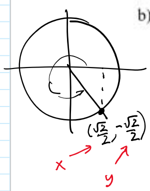
Primary Ratios	***Reciprocal Ratios***
$\sin\theta = \frac{y}{1} = y$	$\csc\theta = \frac{1}{y}$
$\cos\theta = \frac{x}{1} = x$	$\sec\theta = \frac{1}{x}$
$\tan\theta = \frac{y}{x}$	$\cot\theta = \frac{x}{y}$

More Exact Values - Use the unit circle to find each ***exact value – no calculator!***



a) $\cos(\pi) = -1$ $\quad\sin(\pi) = 0$ $\quad\tan(\pi) = \frac{y}{x} = \frac{0}{-1} = 0$

$\sec(\pi) = \frac{1}{-1} = -1$ $\quad\csc(\pi) = \frac{1}{0} =$ undefined $\quad\cot(\pi) = \frac{x}{y} = \frac{-1}{0} =$ undefined

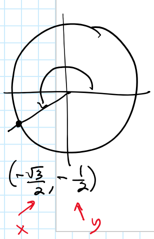


b) $\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $\quad\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ $\quad\tan\left(\frac{7\pi}{4}\right) = \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -\frac{\sqrt{2}}{2} \cdot \frac{2}{\sqrt{2}} = \frac{-\cancel{2\sqrt{2}}}{\cancel{2\sqrt{2}}} = -1$

$\sec\left(\frac{7\pi}{4}\right) = \frac{2}{\sqrt{2}}$ OR $\frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\cancel{2}\sqrt{2}}{\cancel{2}} = \sqrt{2}$

$\csc\left(\frac{7\pi}{4}\right) = -\frac{2}{\sqrt{2}}$

$\cot\left(\frac{7\pi}{4}\right) = \frac{\frac{\sqrt{2}}{2} \leftarrow x}{-\frac{\sqrt{2}}{2} \leftarrow y} = \frac{\sqrt{2}}{2} \cdot \frac{2}{-\sqrt{2}} = \frac{\cancel{2\sqrt{2}}}{-\cancel{2\sqrt{2}}} = -1$



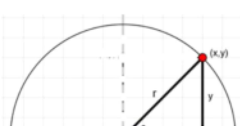
c) $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ $\quad\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$ $\quad\tan\left(\frac{7\pi}{6}\right) = \frac{y}{x} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$ Don't stop yet! $= -\frac{1}{2} \cdot \frac{2}{-\sqrt{3}} = \frac{-\cancel{2}}{-\cancel{2}\sqrt{3}} = \boxed{\frac{1}{\sqrt{3}}}$

$\sec\left(\frac{7\pi}{6}\right) = \frac{2}{-\sqrt{3}} = \frac{-2}{\sqrt{3}} = -\frac{2}{\sqrt{3}}$

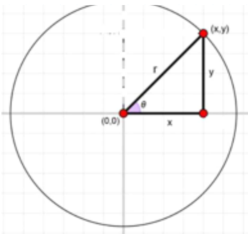
$\csc\left(\frac{7\pi}{6}\right) = \frac{-2}{1} = \boxed{-2}$

$\cot\left(\frac{7\pi}{6}\right) = \frac{\sqrt{3}}{1} = \boxed{\sqrt{3}}$

Sometimes the radius is not 1. Here are the ratio definitions that work for any *r* value.



Primary Ratios	***Reciprocal Ratios***
$\sin\theta = \frac{y}{r}$	$\csc\theta = \frac{r}{y}$
$\cos\theta = \frac{x}{\ }$	$\sec\theta = \frac{r}{\ }$



$\sin\theta = \frac{y}{r}$ $\qquad \csc\theta = \frac{r}{y}$

$\cos\theta = \frac{x}{r}$ $\qquad \sec\theta = \frac{r}{x}$

$\tan\theta = \frac{y}{x}$ $\qquad \cot\theta = \frac{x}{y}$

Know these definitions!

Example

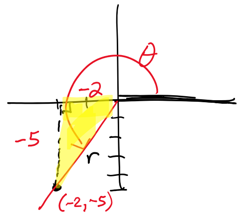
The terminal arm of a standard position angle θ contains the point $(-2,-5)$. Find the value of all six trigonometric ratios for angle θ. You do not need to find the size of angle θ. Leave answers in ***exact fractional form***.

The angle terminates in quadrant 3

$x =$ -2

$y =$ -5

$r =$ $\sqrt{29}$



1) plot point
2) connect it to (0,0)
3) make a $\triangle$, by connecting the point to the x-axis.

$x^2+y^2=r^2$

$(-2)^2+(-5)^2=r^2$

$4+25=r^2$

$29=r^2$

$\sqrt{29}=r$

ALWAYS positive, leave in exact form

$\sin\theta = \frac{y}{r} = \boxed{\frac{-5}{\sqrt{29}}}$ $\qquad \csc\theta = \boxed{\frac{\sqrt{29}}{-5}}$

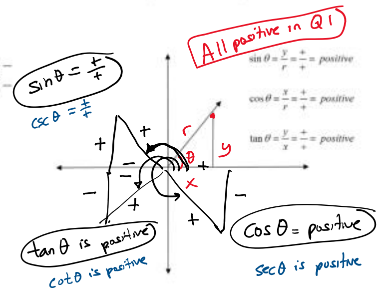
$\cos\theta = \frac{x}{r} = \boxed{\frac{-2}{\sqrt{29}}}$ $\qquad \sec\theta = \boxed{\frac{\sqrt{29}}{-2}}$

$\tan\theta = \frac{y}{x} = \frac{-5}{-2} = \boxed{\frac{5}{2}}$ $\qquad \cot\theta = \boxed{\frac{2}{5}}$

Finding the Signs of the Trigonometric Ratios

How can we predict whether a specific trigonometric ratio will be positive or negative?

r - positive in every quadrant
x - depends on the quadrant, can be + or –
y - depends on the quadrant, can be + or –



Example

Given the information below, decide in which quadrant (or quadrants) angle θ can terminate.

a) $\tan\theta < 0$

$\tan\theta > 0$ in Q1, Q3

$\boxed{\tan\theta < 0 \text{ in } Q2, Q4}$

b) ~~$\csc > 0$~~ $\csc\theta > 0$

$\sin\theta > 0$

$\boxed{Q1, Q2}$

c) $\sin\theta < 0$ *and* $\cot\theta > 0$

$\tan\theta > 0$

S A T C

$\boxed{Q3}$

d) $\sec\theta > 0$ *and* $\tan\theta < 0$

$\cos\theta > 0$

$\boxed{Q4}$

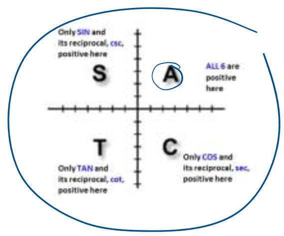
Example

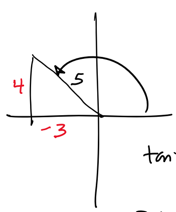
Find the value of $\sec\theta$, if we know $\tan\theta = -\frac{4}{3}$ and $\sin\theta > 0$.

tells us θ can be in Q2, Q4

tells us θ can be in Q1, Q2

$\Rightarrow$ Q2





$\tan\theta = -\frac{4}{3} \leftarrow y, \leftarrow x$

Find r-value: $x^2 + y^2 = r^2$

$(-3)^2 + (4)^2 = r^2$

$9 + 16 = r^2$

$25 = r^2$, $\boxed{r=5}$

$\cos\theta = \frac{x}{r} = \frac{-3}{5}$

$\boxed{\sec\theta = \frac{5}{-3}}$

Approximate Values

For angles not related to special angles, calculators can give us accurate approximations.

Try

Evaluate each of the following ratios correct to 4 decimal places, using a calculator.

a) $\tan(-65^\circ) =$ -2.1445

b) $\sec 417^\circ =$ $\frac{1}{\cos 417^\circ}$

$\doteq 1.8361$

c) $\cot\left(\frac{3\pi}{5}\right) =$ $\frac{1}{\tan\left(\frac{3\pi}{5}\right)}$

use radian mode

$= -0.3249$

d) $\cos(4) =$ -0.6536

use radian mode

e) $\sin(-200^\circ) =$ 0.3420

use degree mode

f) $\csc\left(\frac{4\pi}{7}\right) =$ $\frac{1}{\sin\left(\frac{4\pi}{7}\right)}$

$= 1.0257$

Use the correct mode – either degrees or radians, matching the angle's units.

Be careful when evaluating reciprocal ratios

Use the correct mode – either degrees or radians, matching the angle's units.

Be careful when evaluating reciprocal ratios.
Don't take the reciprocal of the angle!
Instead, use these relationships:

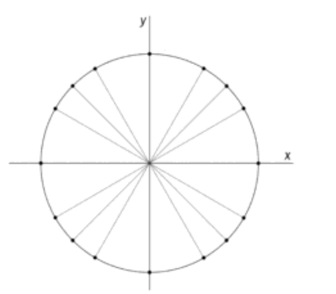
$$\csc(angle) = \frac{1}{\sin(angle)}$$

$$\cot(angle) = \frac{1}{\tan(angle)}$$

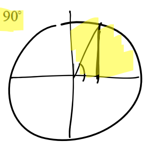
$$\sec(angle) = \frac{1}{\cos(angle)}$$

TB 4.3 p 201: 1-2(acegik), 3ace, 6ace, 9ace, 12ac

Use your unit circle to solve these equations:

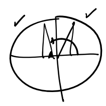


1. $\sin\theta = \frac{\sqrt{3}}{2}$, $0^\circ \le \theta < 90^\circ$



60°

2. $\sin\theta = \frac{\sqrt{3}}{2}$, $0^\circ \le \theta < 360^\circ$

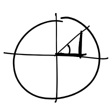


$\boxed{60^\circ}$

$180^\circ - 60^\circ$

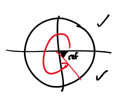
$= \boxed{120^\circ}$

3. $\cos\theta = \frac{\sqrt{2}}{2}$, $0^\circ \le \theta < 90^\circ$



$\boxed{45^\circ}$

4. $\cos\theta = \frac{\sqrt{2}}{2}$, $0^\circ \le \theta < 360^\circ$



Q_1 answer $= \boxed{45^\circ}$

Q_4 answer $=$

$360^\circ - 45^\circ$

$= \boxed{315^\circ}$

4.4 Solving Trigonometric Equations

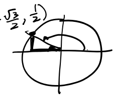
We now know how to find trigonometric ratio values for any angle. Sometimes we must settle for an approximation, other times we can give an exact value.

For example:

$\sin 47^\circ = 0.7314$ (calculator needed)

$\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ Unit circle

$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$



Now we look at the opposite situation. Given the value of a trigonometric ratio, how do we find out the size of the ***angle***? This is a process that works well:

Isolate – Decide – Get Reference Angle - Solve

See page 21.

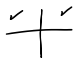
Examples Solve algebraically. If possible, give exact answers. Otherwise, give answers correct to one decimal place.

a) $\sin\theta - 0.8 = 0$, for $0^\circ \le \theta < 360^\circ$ ← tells us answers must be in degrees.

1) isolate function

$\sin\theta - 0.8 = 0$
$+0.8 \quad +0.8$

$\sin\theta = 0.8$



2) decide which method
" " quadrants

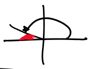
We'll use calculator.
Q1, Q2

$\sin\theta = 0.8$

Where is it that sine is positive?

3) reference angle

$\theta_R = \sin^{-1}(0.8)$
$\doteq 53.1^\circ$



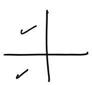
4) get answers

$\boxed{Q_1 = 53.1^\circ}$

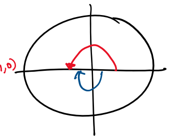
$Q_2 = 180^\circ - \theta_R = 180^\circ - 53.1^\circ = \boxed{126.9^\circ}$

b) $\cos\theta = -1$, for $-\pi \le \theta < 2\pi$

decide – use unit circle
– answers in Q2, Q3



(−1, 0)



answer: $\boxed{\pi}$

another answer is the coterminal angle:

$\pi - 2\pi$

$= \boxed{-\pi}$

Isolate – Decide – Get Reference Angle - Solve

1) Isolate the trigonometric term. If it uses cot, sec, or csc, take the ***reciprocal*** of both sides of the equation to get a simpler-to-solve version of the equation.

2) Decide whether the equation can be solved using

- special angles on the unit circle. Look for: $0, \pm 1, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{3}}{2}$
- the $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ button on the calculator
- OR, cannot be solved

3) Determine in which quadrants answers will be found.

4) Find the reference angle and use it to find all the solutions in the given domain. Use the same units (either degrees or radians) as shown in the question's domain.

radian mode

c) $3\cos\theta - 12 = 0$, for $0 \le \theta < 2\pi$

$\frac{3\cos\theta}{3} = \frac{12}{3}$

$\cos\theta = \frac{4}{1}$ ← adj, ← hyp

calculator; Q1, Q4

$\theta_R = \cos^{-1}(4)$

$\boxed{\text{no solution}}$

d) $3\tan\theta - 3 = 0$, for $0 \le \theta < 2\pi$

$\frac{3\tan\theta}{3} = \frac{3}{3}$

$\tan\theta = 1 = \frac{y}{x}$

$\theta_R = \pi/4$

$Q_1 = \boxed{\pi/4}$

$Q_3 = \pi + \theta_R$

$\frac{4}{4}\pi + \pi/4$

$= \boxed{\frac{5\pi}{4}}$

e) $3\sec\theta - 10 = 0$, for $0^\circ \le \theta < 720^\circ$

$3\sec\theta = 10$

$\sec\theta = \frac{10}{3}$

$\frac{1}{\cos\theta} = \frac{10}{3}$

(take reciprocal of both side of the equation)

$\cos\theta = \frac{3}{10}$

decide: calculator Q1, Q4

$\theta_R = \cos^{-1}\left(\frac{3}{10}\right)$

$\doteq 72.5^\circ$

Answers

$Q_1 = \boxed{72.5^\circ}$

$Q_4 = 360^\circ - \theta_R$

$= \boxed{287.5^\circ}$

get coterminals because the domain goes to 720°

$72.5^\circ + 360^\circ = \boxed{432.5^\circ}$

$287.5^\circ + 360^\circ = \boxed{647.5^\circ}$

Whiteboards

Solve these two trigonometric equations. If exact answers are possible, give those. Otherwise, round values correct to 2 decimal places.

For each equation answer:

- In which quadrants will the answers be?
- What is the size of the reference angle?
- What are the solutions?

1. $8\sin\theta - 4 = 0,\ 0 \le \theta < 2\pi$

$8\sin\theta = 4$

$\sin\theta = \frac{4}{8}$

$\sin\theta = \frac{1}{2}$

$\theta_R = \pi/6$

$\boxed{\pi/6,\ \frac{5\pi}{6}}$

$Q_2 = \pi - \theta_R$

$= \frac{6}{6}\pi - \pi/6$

$= 5\pi/6$

2. $5\tan\theta - 1 = 0,\ 0^\circ \le \theta < 360^\circ$

$5\tan\theta = 1$

$\tan\theta = \frac{1}{5}$

$\theta_R = \tan^{-1}\left(\frac{1}{5}\right)$

$\theta_R \doteq 11.3^\circ$

$\boxed{Q_1 = 11.3^\circ}$

$Q_3 = 180^\circ + \theta_R = \boxed{191.3^\circ}$

Which of these trigonometric equations have NO SOLUTION?

1. $\sin\theta = -2$
2. $\cos\theta = 0.3$
3. $\tan\theta = 0.9$
4. $\tan\theta = 15$
5. $\sin\theta = -0.5$
6. $\cos\theta = -1.4$

Isolate – Decide – Get Reference Angle - Solve

1) Isolate the trigonometric term. If it uses cot, sec, or csc, take the ***reciprocal*** of both sides of the equation to get a simpler-to-solve version of the equation.

2) Decide whether the equation can be solved using

- special angles on the unit circle. Look for: $0, \pm 1, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{3}}{2}$
- the $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ button on the calculator
- OR, cannot be solved

3) Determine in which quadrants answers will be found.

4) Find the reference angle and use it to find all the solutions in the given domain. Use the same units (either degrees or radians) as shown in the question's domain.

Examples Solve algebraically. If possible, give exact answers. Otherwise, give answers correct to one decimal place.

a) $\sin\theta = -0.8$, for $0^\circ \le \theta < 360^\circ$

$\theta_R = \sin^{-1}(+0.8)$
$\doteq 53.1^\circ$ use this to get answers

$Q_3 = 180^\circ + \theta_R = \boxed{233.1^\circ}$
$Q_4 = 360^\circ - \theta_R = \boxed{306.9^\circ}$

If we calculate $\sin^{-1}(-0.8)$, the calculator gives us a negative answer. We don't want this, because the domain asks for only positive answers.
To avoid this problem, give the calculator the trigonometric ratio as a positive quantity. This guarantees the calculator will give us the ***reference angle***, which will be positive.

b) $5\cos\theta - 2 = 2\cos\theta - 4$, for $0^\circ \le \theta < 720^\circ$

$-2\cos\theta \quad -2\cos\theta$

$3\cos\theta - 2 = -4$
$+2 \quad +2$

$3\cos\theta = -2$

$\cos\theta = -\frac{2}{3}$

Calculator
Q2, Q3

$\theta_R = \cos^{-1}\left(+\frac{2}{3}\right)$
$\doteq 48.2^\circ$

Q_2 answer $= 180^\circ - \theta_R$
$= \boxed{131.8^\circ}$

Q_3 answer $= 180^\circ + \theta_R$
$= \boxed{228.2^\circ}$

Also (domain is $0 \le \theta < 720^\circ$)
get coterminals:
$131.8^\circ + 360^\circ = \boxed{491.8^\circ}$
$228.2^\circ + 360^\circ = \boxed{588.2^\circ}$

General Solution

How many solutions a trigonometric equation has depends on the domain specified in the question. When the domain is all real numbers, there are infinitely many solutions. This is called the ***general solution***.

Example Solve algebraically. If possible, give exact answers. Otherwise, give answers correct to one decimal place.

a) $3\cos\theta - 1 = 0$ ***general solution*** in degree measure.

$3\cos\theta = 1$

$\cos\theta = \frac{1}{3}$

$\theta_R = \cos^{-1}\left(\frac{1}{3}\right)$

$\doteq 70.5°$

$Q_1 = 70.5°$

$Q_4 = 360° - \theta_R$

$= 289.5°$

General solution:

$70.5° + 360°n,\ n \in I$

$289.5° + 360°n,\ n \in I$

b) $\cot\theta + 5 = 0$, ***general solution*** in radian measure.

$\cot\theta = -5$

$\frac{1}{\tan\theta} = -5$

$\tan\theta = -\frac{1}{5}$

Use calculator, Q2 and Q4

$\theta_R = \tan^{-1}\left(+\frac{1}{5}\right)$

$\doteq 0.2$

Q_2 answer $= \pi - \theta_R$

$\doteq 2.9$

Q_4 answer $= 2\pi - \theta_R$

$\doteq 6.1$

General

$2.9 + 2\pi n,\ n \in I$

$6.1 + 2\pi n,\ n \in I$

Here's how to write the ***general solution***:

- list each answer θ, found in one full rotation, separately
- to each answer add on the appropriate amount, either

$+2\pi n,\ n \in I$ or $+360°n,\ n \in I$

For equations using *tangent* or *cotangent*, we find that in one full rotation the two solutions are spaced exactly π or $180°$ apart. Because of this, we can just write the first solution, and add onto it:

$+\pi n,\ n \in I$ or $+180°n,\ n \in I$

Example

Suppose that for a certain equation, we are all told its solutions for $0° \le \theta < 360°$ are $\theta = 20°$ and $\theta = 160°$. What is the ***general solution***?

General solution:

$20° + 360°n,\ n \in I$

$160° + 360°n,\ n \in I$

Solving Second-Degree Trigonometric Equations
When we solve equations with an exponent we usually start by factoring.
For example, solve: $2x^2 - 1 = x$

$AC = -2$

$-2, 1$

$2x^2 - x - 1 = 0$

Factor!

$2x^2 - 2x + x - 1 = 0$

$2x(x-1) + 1(x-1) = 0$

$(x-1)(2x+1) = 0$

$x - 1 = 0$ $2x + 1 = 0$

$x = 1$ $2x = -1$

$x = -\frac{1}{2}$

https://www.youtube.com/watch?v=gENVB6tjq_M

Example
Solve algebraically. If possible, give exact answers. Otherwise, give answers correct to one decimal place.

$$2\tan^2\theta - 1 = \tan\theta \text{, for } 0^\circ \le \theta < 360^\circ$$

We have to start these by getting everything to one side, then factoring. Once we have factored, we set each factor equal to zero. This means we will have two equations to solve.

(4.4) TB p 211: 2-4, 5ace, 7 all
Trigonometry Practice #4

Coming up
- **Starting Chapter 5 next class (trigonometric graphs)**
- **Chapter 4 Hand-in due Thursday, May 18**
- **No class on Monday, May 22 (Victoria Day weekend)**
- **Test 3 on Tuesday, May 23 (sections 4.2-5.4, omitting 5.3).**

More Practice
Worksheet: More Chapter 4 Review, found on website (with solutions)
Worksheet: Trig Practice #4, on website (with solutions)