Class_08 Oct 4 - Unit Circle and Special Angles

Wednesday, September 28, 2022 9:39 PM

Tonight's Class

Grab a white board, pen and eraser
Returning Unit 1 Test, rewrite sign-up
4.1 Arc Length
4.2 Unit Circle and Special Angles
4.3 Trigonometric Ratios

WB - random groups

The graph of y = f(x) is transformed to produce y = g(x).
What is the equation of the transformed graph?

Strategy - figure out what you think it is. Then make a table of key points from the original graph and actually perform your transformations to it. See whether or not you are right.

Check-in

$\pi \ radians = 180^\circ$

1. $2\pi = 360^\circ$

2. $\frac{\pi}{3} = 60^\circ$

3. $\frac{\pi}{2} = 90^\circ$

4. $\frac{\pi}{6} = 30^\circ$

5. $\frac{3\pi}{2} = 270^\circ$

$\frac{3}{2} \times \frac{180^\circ}{1}$

$= \frac{540^\circ}{2}$

$= 270^\circ$

6. $\frac{\pi}{4} = 45^\circ$

7. $270^\circ =$

$270^\circ \times \frac{\pi}{180^\circ} = \frac{27\pi}{18}$

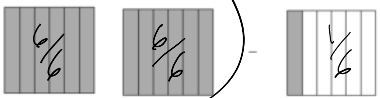
$= \frac{3\pi}{2}$

8. $45^\circ = \frac{\pi}{4}$

Working with Radians in Fraction Form

Because π radians is the size of a straight angle (half a rotation), we end up working a lot with angles written as fractional parts of π. Let's review adding/subtracting fractions.

$2 - \frac{1}{6} =$ $= \frac{12}{6}$



$\frac{2}{1}\cdot\frac{6}{6} - \frac{1}{6} = \frac{12}{6} - \frac{1}{6}$

$= \frac{11}{6}$

Try

a) $1 + \frac{1}{4} =$ $\frac{4}{4} + \frac{1}{4} = \frac{5}{4}$

b) $2 - \frac{1}{3} =$ $\frac{6}{3} - \frac{1}{3} = \frac{5}{3}$

c) $1 - \frac{1}{6} =$ $\frac{5}{6}$

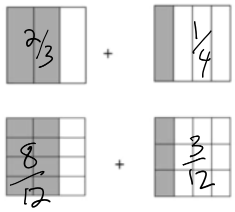
d) $2 - \frac{1}{4} =$ $\frac{7}{4}$

e) $1 - \frac{1}{4} =$ $\frac{3}{4}$

f) $1 + \frac{1}{3} =$ $4/3$

Different Denominators

$\frac{2}{3} + \frac{1}{4} =$



$\frac{2}{3}\cdot\frac{4}{4} + \frac{1}{4}\cdot\frac{3}{3}$

$= \frac{8}{12} + \frac{3}{12}$

$= \frac{11}{12}$

Try

a) $\frac{1}{2} + \frac{1}{3} =$ $\frac{3}{6} + \frac{2}{6}$

$= \frac{5}{6}$

b) $\frac{7}{6} + \frac{1}{4} =$ $\frac{14}{12} + \frac{3}{12}$

$\frac{34}{24} \longrightarrow = \frac{17}{12}$

c) $\frac{3}{5} + \frac{1}{7} =$ $\frac{21}{35} + \frac{5}{35}$

$= \frac{26}{35}$

Is π in the fraction? It still works the same way!

Try

a) $\pi + \frac{\pi}{6} =$

$= \frac{6\pi}{6} + \frac{\pi}{6}$

$= \frac{7\pi}{6}$

b) $2\pi - \frac{\pi}{2} =$

$\frac{4\pi}{2} - \frac{\pi}{2}$

$= \frac{3\pi}{2}$

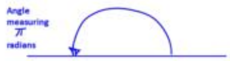
c) $\frac{3\pi}{4} + 2\pi \cdot \frac{4}{4}$

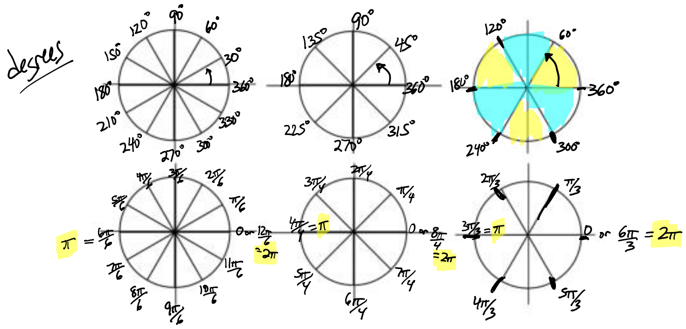
$= \frac{3\pi}{4} + \frac{8\pi}{4}$

$= \frac{11\pi}{4}$

Standard-Position Angles in Radian Measure

A straight angle measures π radians. This helps us when sketching angles that are fractions involving π.



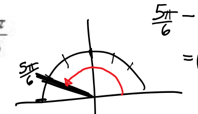


For each angle below:

- Draw the angle in standard position.
- List two angles that are coterminal to the given angle. Use radians, not degrees!

a) $\frac{5\pi}{6}$

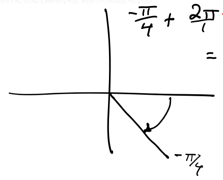
this tells us the size of the little piece



$\frac{5\pi}{6} - \frac{12\pi}{6} = -\frac{7\pi}{6}$

$\frac{5\pi}{6} + \frac{2\pi}{1}\cdot\frac{6}{6} = \frac{5\pi}{6} + \frac{12\pi}{6} = \frac{17\pi}{6}$

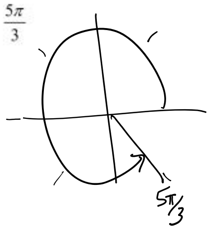
b) $-\frac{\pi}{4}$



$-\frac{\pi}{4} + \frac{2\pi}{1}\cdot\frac{4}{4} = -\frac{\pi}{4} + \frac{8\pi}{4} = \frac{7\pi}{4}$

$-\frac{\pi}{4} - \frac{8\pi}{4} = -\frac{9\pi}{4}$

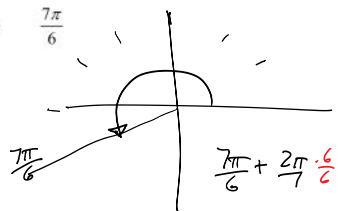
c) $\frac{5\pi}{3}$



$\frac{5\pi}{3} + 2\pi\cdot\frac{3}{3} = \frac{5\pi}{3} + \frac{6\pi}{3} = \frac{11\pi}{3}$

$\frac{5\pi}{3} - \frac{6\pi}{3} = -\frac{\pi}{3}$

d) $\frac{7\pi}{6}$



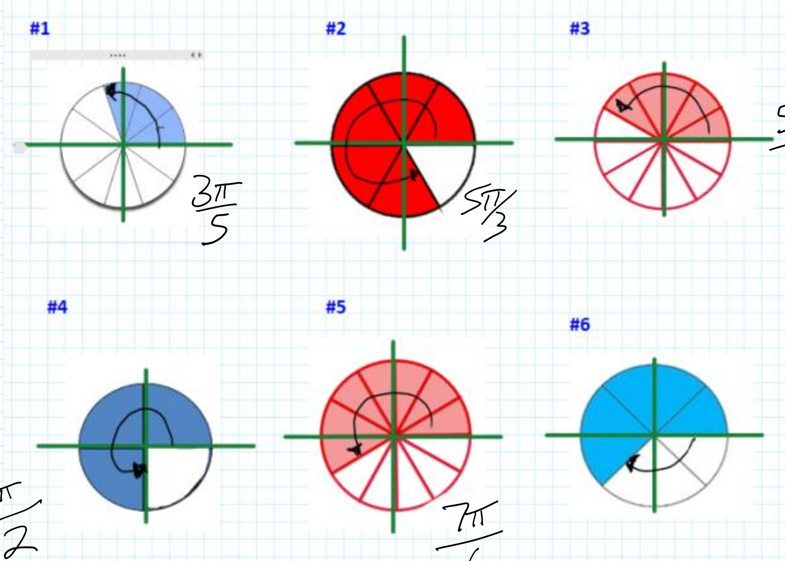
$\frac{7\pi}{6} + \frac{2\pi}{1}\cdot\frac{6}{6} = \frac{7\pi}{6} + \frac{12\pi}{6} = \frac{19\pi}{6}$

$\frac{7\pi}{6} - \frac{12\pi}{6} = \boxed{-\frac{5\pi}{6}}$

TB p 175: 6, 7ab, 8ac, 9ab

Quick Check-in

What is the size of each angle, in radian measure?



#1: $\frac{3\pi}{5}$

#2: $\frac{5\pi}{3}$

#3: $\frac{5\pi}{6}$

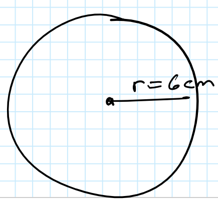
#4: $\frac{3\pi}{2}$

#5: $\frac{7\pi}{6}$

#6: $\frac{-3\pi}{4}$

I don't understand why some people use fractions instead of decimals.

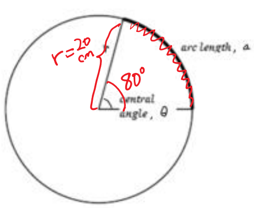
It's pointless.



$$C = 2\pi r$$
$$= 2\pi(6)$$
$$= 12\pi$$
$$\doteq 37.7 \text{ cm}$$

Arc Length

Suppose we have a circle with radius = 20 cm. If we mark off a central angle measuring 80°, what arc length (length along the circle's circumference) does the angle cut off?



$$\frac{80^\circ}{360^\circ} = \frac{\text{arc length}}{\text{Circumference}}$$

$$C = 2\pi r$$
$$= 2\pi(20)$$
$$= 40\pi$$

$$40\pi \cdot \left(\frac{80^\circ}{360^\circ}\right) = \left(\frac{a}{\cancel{40\pi}}\right)\cancel{40\pi}$$

$$a = \frac{80(40\pi)}{360}$$

$$a \doteq 27.9 \text{ cm}$$

$$\boxed{a = r\theta}$$

Formula that is faster:

$$2\pi r\left(\frac{\theta^\circ}{360^\circ}\right) = \left(\frac{a}{\cancel{2\pi r}} \times\right)\cancel{2\pi r}$$

$$a = \frac{\theta}{360} \cdot 2\pi r$$

360

$$\boxed{a = r\theta}$$

if angle is in radians

$$a \doteq 27.9 \text{ cm}$$

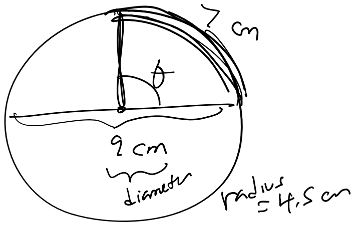
approximate

$$\boxed{a = r\theta \cdot \frac{\pi}{180^\circ}}$$

$$a = \frac{\theta}{\cancel{360}_{180}} \cdot \frac{\cancel{2}^{1}\pi r}{1}$$

$$a = \theta r \left(\frac{\pi}{180}\right)$$

Suppose we have another circle, this one with diameter 9 cm. If we know that an arc measuring 7 cm is subtended by a central angle, what is the measure of that central angle, in radians? What is the measure of the central angle, in degrees?



$$a = r\theta$$

$$\frac{7}{4.5} = \frac{\cancel{4.5}\,\theta}{\cancel{4.5}}$$

$$\theta = \frac{7}{4.5} = 1.\overline{5} \text{ radians}$$

$$= 1.\overline{5} \times \left(\frac{180^\circ}{\pi}\right)$$

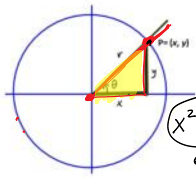
$$\doteq 89.1^\circ$$

TB p 176: 12ac, 13

WB Groups - Arc Length

4.2 The Unit Circle

A circle is the set of all points that are a certain distance, ***radius***, from a given point, the center. Using the Pythagorean Theorem, we can get an equation for a circle.



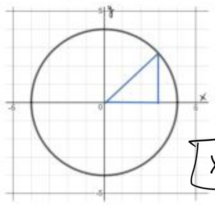
The equation for a circle with center $(0,0)$ and radius r is:

$$x^2 + y^2 = r^2$$

$\boxed{x^2+y^2=r^2}$
equation of circle, with radius $= r$
Centre $= (0,0)$

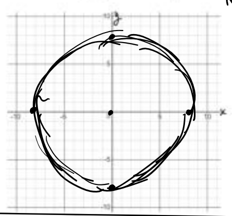
Try

a) Find the equation of this circle.



$x^2+y^2=r^2$
$r=4$
$\boxed{x^2+y^2=16}$

b) Sketch the graph of $x^2 + y^2 = 64$

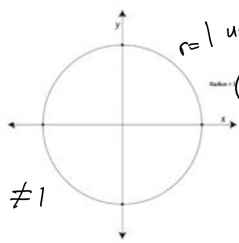


$r^2=64$
$\Rightarrow r=8$

Unit Circle

If we choose $r = 1$, we get a circle with radius 1 unit in length. This is called the ***unit circle***, and its equation is $x^2 + y^2 = 1$.

$x^2+y^2=1^2$



$r = 1$ unit,
Unit circle

a) Is the point $(0.6, 0.4)$ on the unit circle?

x y

Substitute:

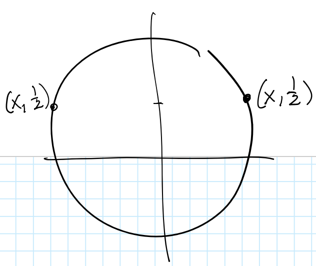
$x^2+y^2=1$
$(0.6)^2+(0.4)^2 = 0.52 \neq 1$

no

b) The point below is on the unit circle. Use the unit circle equation, $x^2 + y^2 = 1$, to find the value of the unknown coordinate.

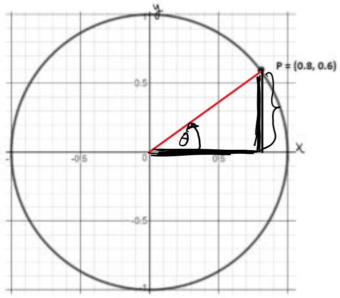
$\left(x, \frac{1}{2}\right)$

$$x^2+y^2=1$$
$$x^2+\left(\frac{1}{2}\right)^2=1$$
$$x^2+\frac{1}{4}=1$$
$$x^2=\frac{4}{4}1-\frac{1}{4}$$
$$\sqrt{x^2}=\pm\sqrt{\frac{3}{4}}$$



$$x=\pm\sqrt{\frac{3}{4}}=\frac{\pm\sqrt{3}}{\sqrt{4}}=\frac{\pm\sqrt{3}}{2}$$

On the unit circle below, we have a point P, with coordinates (0.8, 0.6). We draw a line segment connecting P to the origin, (0, 0). This radius and the x-axis form a standard position angle, which we call θ. Because this is an accurate drawing, we could use a protractor and get the size of angle θ – it is about 36.87°. By drawing in a line segment that connects P to the x-axis, we create a right-triangle, with the right-angle on the x-axis.



From the diagram, we get:

$\cos\theta = \frac{adj}{hyp} = \frac{0.8}{1} = 0.8$

$\sin\theta = \frac{opp}{hyp} = \frac{0.6}{1} = 0.6$

$\tan\theta = \frac{opp}{adj} = \frac{y}{x}$

Using the calculator, we get:

$\cos 36.87^\circ = 0.799\ldots \doteq 0.8$

$\sin 36.87^\circ = 0.6000042 \doteq 0.6$

Let $P(\theta) = (x, y)$ be the point where the terminal arm of a standard-position angle θ intersects the **unit circle**. Then we know:

- the x-coordinate's value is equal to the cosine of the angle $\quad x = \cos\theta$
- the y-coordinate's value is equal to the sine of the angle $\quad y = \sin\theta$

We now have a way to find sine and cosine values for ANY angle, including:

- negative angles
- 0°
- angles larger than 90°

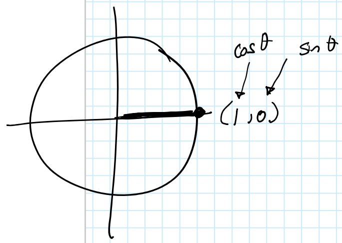
http://www.malinc.se/math/trigonometry/unitcircleen.php

Using the triangle definitions (SOHCAHTOA) for those types of angles doesn't really make sense. For example, what would be the adjacent, opposite, and hypotenuse lengths for an angle measuring 0°?

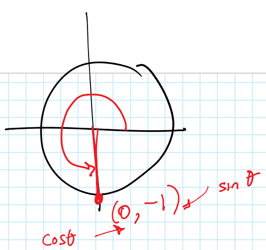
Try – NO calculator. *(You don't need it! You can figure them out yourself!)*

$P(0^\circ) = (1, 0)$ $\cos(0^\circ) =$ x-coordinate $= 1$ $\sin(0^\circ) = 0$

point where 0° intersects the unit circle

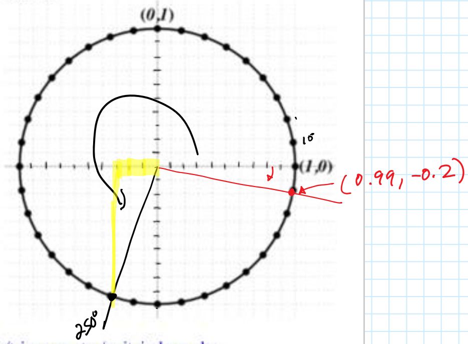


$P\left(\frac{3\pi}{2}\right) = (0, -1)$ $\sin\left(\frac{3\pi}{2}\right) = -1$ $\cos\left(\frac{3\pi}{2}\right) = 0$



Finding Approximate Values of Trigonometric Ratios

Estimate each value using the graph at right. Compare with the calculator answer, correct to 4 decimal places.



a) $\cos 250^\circ =$ −0.3 $\qquad \sin 250^\circ =$

x-coordinate

b) $\cos 500^\circ =$ $\qquad \sin 500^\circ =$

c) $\cos(-10^\circ) =$ 0.99 $\qquad \sin(-10^\circ) =$ −0.2

http://www.malinc.se/math/trigonometry/unitcircleen.php

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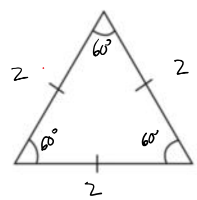
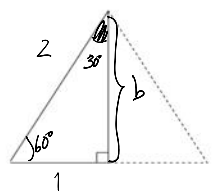
Special Triangle Angles

Besides the quadrantal angles, there are some other angles for which we can find ***exact*** coordinates for $P(\theta)$. These angles relate to special triangles.

Remember special triangles?



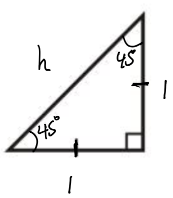


$b^2 + 1^2 = 2^2$

$b^2 + 1 = 4$

$b^2 = 3$

$b = \sqrt{3}$



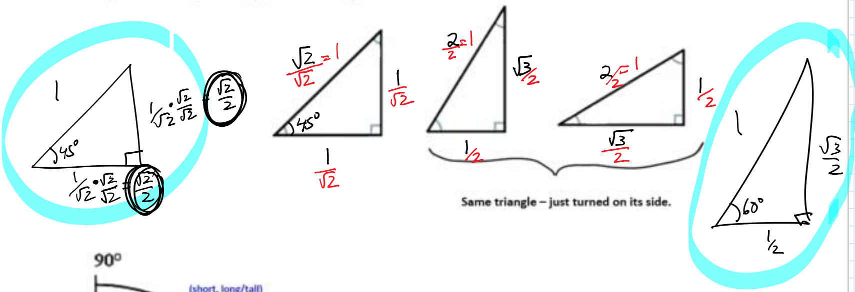
$1^2 + 1^2 = h^2$

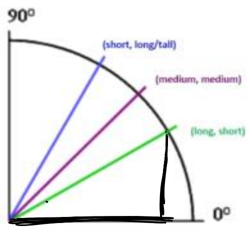
$1 + 1 = h^2$

$2 = h^2$

$h = \sqrt{2}$

Let's use those triangle angles in the unit circle setting. We need to adjust the size of the triangles, making their hypotenuse length = 1.





Short side length = $\frac{1}{2}$ (0.5)

Medium side length = $\frac{\sqrt{2}}{2}$ (0.707)

Tall/long side length = $\frac{\sqrt{3}}{2}$ (0.866)

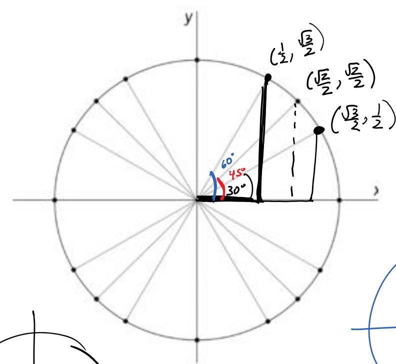
We know that the shortest side of a triangle is across from its smallest angle. This helps us label the coordinates correctly for different angles.

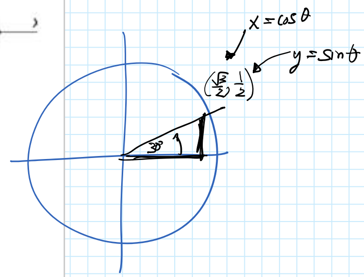
Try – NO calculator. Get the ***exact values***.

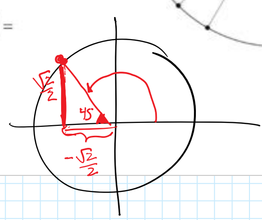
$\cos(30^\circ) =$ x-coordinate $= \frac{\sqrt{3}}{2}$ $\qquad \sin(30^\circ) = \frac{1}{2}$

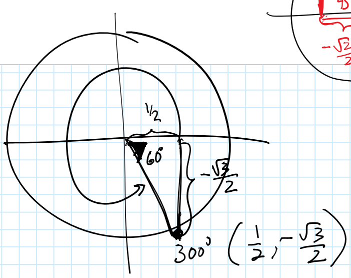
$\sin(135^\circ) = \frac{\sqrt{2}}{2}$ $\qquad \cos(135^\circ) = -\frac{\sqrt{2}}{2}$ x-coord

$\cos(300^\circ) = \frac{1}{2}$ $\qquad \sin(300^\circ) = -\frac{\sqrt{3}}{2}$









Unit Circle Worksheet - fill in the front side only

TB p 186: 1c, 2ace, 3ac, 4, 5

For next class
Practice the ideas in 4.1-4.3
- **(4.2) p 186: 1c, 2ace, 3ac, 4, 5**
- **(4.3) p 201: 1-2(acegik), 3ace, 6ace, 9ace, 10all, 11 all, 12ac**
- **Trigonometry Practice #1-3 (worksheets, posted on website)**

Start working on Ch 4 Hand-in Assignment, #1-10

Unit Circle videos
- https://www.youtube.com/watch?v=ao4EJzNWmK8
  = 4 minute video, first quadrant only

- https://www.youtube.com/watch?v=c819bGfH8FA
    - **= 12 minute video**