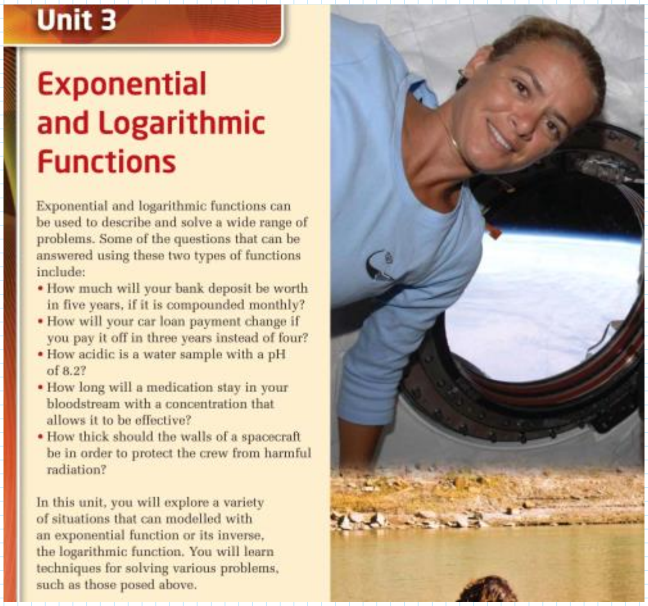
Class_15 May 30 - Exponential Functions

Monday, May 29, 2023 2:45 PM

Tonight's Class:

- **Collecting Chapter 6 Hand-in Assignment**
- **Returning Test 4**
- **7.1 Characteristics of Exponential Functions**
- **7.2 Transforming Exponential Functions**

Textbook, page 330



Unit 3

Exponential and Logarithmic Functions

Exponential and logarithmic functions can be used to describe and solve a wide range of problems. Some of the questions that can be answered using these two types of functions include:

- How much will your bank deposit be worth in five years, if it is compounded monthly?
- How will your car loan payment change if you pay it off in three years instead of four?
- How acidic is a water sample with a pH of 8.2?
- How long will a medication stay in your bloodstream with a concentration that allows it to be effective?
- How thick should the walls of a spacecraft be in order to protect the crew from harmful radiation?

In this unit, you will explore a variety of situations that can modelled with an exponential function or its inverse, the logarithmic function. You will learn techniques for solving various problems, such as those posed above.

Unit 3

In Unit 3 we work with exponential functions, which are functions in this form:

exponential function

- a function of the form $y = c^x$, where c is a constant ($c > 0$) and x is a variable

$y = 2^x$

$y = \left(\frac{1}{2}\right)^x$

~~$y = (-5)^x$~~ no

Note:

- **The base cannot be negative**
- **The base cannot be 0**

~~$y = 0^x$~~

Quick EXPONENTS Review

4^2

$2^3 = 2\times2\times2 = 8$

$2^2 = 2\times2 = 4$

$2^1 = 2 = 2$

$2^0 = \quad = 1$

$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$

$2^{-2} = \frac{1}{2\times2} = \frac{1}{4}$

$2^{-3} = \frac{1}{2\times2\times2} = \frac{1}{8}$

$2^{-7} = \frac{1}{2^7} = \frac{1}{128}$

$\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16}$

Zero Exponent

$(anything)^0 = 1$

except 0^0 = indeterminate

Negative Exponents

$(5)^{-3} = \frac{1}{(5)^{+3}} = \frac{1}{125}$

$\left(\frac{1}{4}\right)^{-2} = \frac{1}{\left(\frac{1}{4}\right)^2} = \frac{1}{\frac{1}{16}} = 1 \div \frac{1}{16} = 1 \times \frac{16}{1} = \boxed{16}$

$\left(\frac{2}{5}\right)^{-2} = \left(\frac{5}{2}\right)^{+2} = \frac{25}{4}$

Faster approach:

$\left(\frac{1}{4}\right)^{-2}$ (reciprocal of fraction) +exponent

$\left(\frac{4}{1}\right)^{+2} = \frac{4^2}{1^2} = 16$

Fractions to a Negative Exponent – Shorter Method

Take reciprocal of the base and change the exponent to a positive exponent.
Evaluate.

$$\left(\frac{5}{8}\right)^{-2} = \left(\frac{8}{5}\right)^{+2} = \frac{64}{25}$$

Chapter 7: Exponential Functions

7.1 Characteristics of Exponential Functions

An exponential function is a function where the ***exponent*** includes a variable, and the ***base*** is larger than zero, not equal to 1. Exponential functions are used to model many real-life situations of change – such as population growth, radioactive decay and compound interest.

For example –
Suppose you greet three people.  3 people

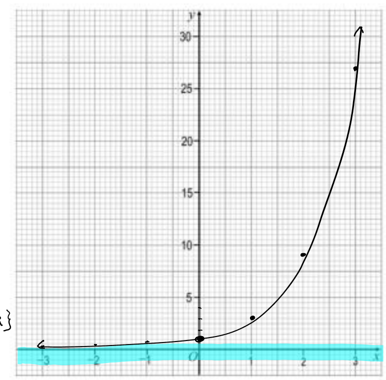
Each person you greeted goes on to greet 3 different people.  9 people

If this pattern continues, you can see that the number of people greeted grows very quickly. 27 people

Consider the function $y = 3^x$.
a) Complete the table, then sketch the graph of $y = 3^x$ on the grid.

$y=3^x$

x	y	
−3	3^{-3}	$= \frac{1}{3^3} = \frac{1}{27}$
−2	3^{-2}	$= \frac{1}{3^2} = \frac{1}{9}$
−1	3^{-1}	$= \frac{1}{3}$
0	$3^0 =$	1
1	$3^1 =$	3
2	$3^2 =$	9
3	$3^3 =$	27



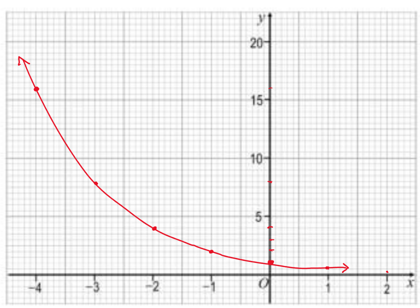
b) State the graph's:
domain $\{x \mid x \in \mathbb{R}\}$
range $\{y \mid y > 0, y \in \mathbb{R}\}$
y-intercept $(0, 1)$
x-intercept none
horizontal asymptote equation $y = 0$

Exponential Growth
(left to right → y-values are increasing)

Example
a) Create a table, then sketch the graph of the exponential function $y = \frac{1}{2}^x$ on the grid.

$y=\left(\frac{1}{2}\right)^x$

x	y	
−4	$\left(\frac{1}{2}\right)^{-4}$	$= \left(\frac{2}{1}\right)^4 = 16$
−3	$\left(\frac{1}{2}\right)^{-3}$	$= \left(\frac{2}{1}\right)^3 = 8$
−2	$\left(\frac{1}{2}\right)^{-2}$	$= \left(\frac{2}{1}\right)^2 = 4$
−1	$\left(\frac{1}{2}\right)^{-1}$	$= \left(\frac{2}{1}\right)^1 = 2$
0	$\left(\frac{1}{2}\right)^0$	$= 1$
1	$\left(\frac{1}{2}\right)^1$	$= \frac{1}{2}$
2	$\left(\frac{1}{2}\right)^2$	$= \frac{1}{4}$

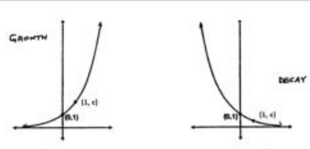


decreasing graph
exponential decay

b) State the graph's:
domain $\{x \mid x \in \mathbb{R}\}$
range $\{y \mid y > 0, y \in \mathbb{R}\}$
y-intercept $(0, 1)$
x-intercept none
horizontal asymptote equation $y = 0$

$y = c^x$, when $c > 1$

$y = c^x$, when $0 < c < 1$



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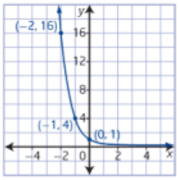
Example 2

Write the Exponential Function Given Its Graph

What function of the form $y = c^x$ can be used to describe the graph shown?



decreasing, $0 < c < 1$

x	y
−2	16
−1	4
0	1

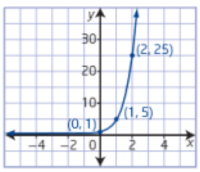
what's happening with y-values? they're being divided by 4

$$y = \left(\frac{1}{4}\right)^x$$

TB p 339

Your Turn

What function of the form $y = c^x$ can be used to describe the graph shown?

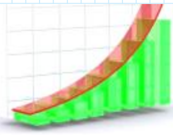


x	y
0	1
1	5
2	25

$$y = 5^x$$

Where Can We See Exponential Growth/Decay?

https://studiousguy.com/exponential-growth-examples/
https://studiousguy.com/exponential-decay-examples/



Exponential growth is a pattern of data that shows a sharp increase over time. The graph of exponentially growing data is generally plotted on a logarithmic scale. There are a number of domains that make use of the concept of exponential growth for research and growth purposes such as biology, finance, mathematics, economics, business, management, etc.

Index of Article (Click to Jump)

Exponential decay describes the process of reduction in the magnitude or value of a particular quantity at a consistent rate over a period of time. In other words, if a value tends to move towards zero rapidly, it is said to be exhibiting an exponential decay. The concept of exponential decay is being utilized by a variety of fields such as finance, biology, chemistry, physics, ecology, archaeology, etc.

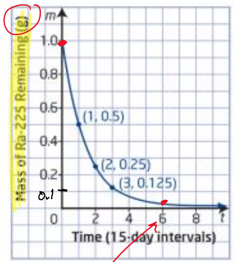
Index of Article (Click to Jump)

TB p 340

Example 3

Application of an Exponential Function

A radioactive sample of radium (Ra-225) has a **half-life** of 15 days. The mass, m, in grams, of Ra-225 remaining over time, t, in 15-day intervals, can be modelled using the exponential graph shown.



a) What is the initial mass of Ra-225 in the sample? What value does the mass of Ra-225 remaining approach as time passes?

t=0, 1.0 g

Approaches 0 g, as time goes by.

b) What are the domain and range of this function?

t ≥ 0; 0 ≤ m ≤ 1

c) Write the exponential decay model that relates the mass of Ra-225 remaining to time, in 15-day intervals.

x	y
0	1
1	0.5
2	0.25
3	0.125

$y = \left(\frac{1}{2}\right)^x$

$m = \left(\frac{1}{2}\right)^t$

d) Estimate how many days it would take for Ra-225 to decay to $\frac{1}{30}$ of its original mass.

≐ 0.03

≃ about 6 15-day intervals ≐ 90 days

TB p 343, #6

Apply

6. Each of the following situations can be modelled using an exponential function. Indicate which situations require a value of $c > 1$ (growth) and which require a value of $0 < c < 1$ (decay). Explain your choices.

a) Bacteria in a Petri dish double their number every hour. — growth

b) The half-life of the radioactive isotope actinium-225 is 10 days. — decay

c) As light passes through every 1-m depth of water in a pond, the amount of light available decreases by 20%. — decay

d) The population of an insect colony triples every day. — growth

TB p 344: #11

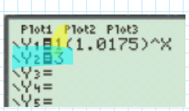
11. Money in a savings account earns compound interest at a rate of 1.75% per year. The amount, A, of money in an account can be modelled by the exponential function $A = P(1.0175)^n$, where P is the amount of money first deposited into the savings account and n is the number of years the money remains in the account.

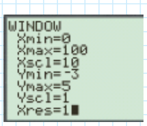
1 + 0.0175

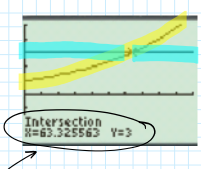
a) Graph this function using a value of $P = \$1$ as the initial deposit.

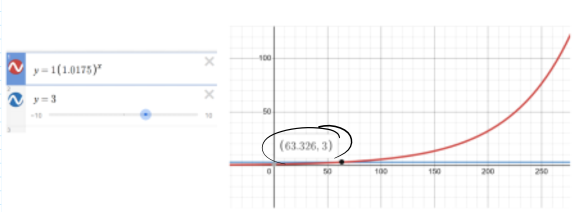
b) Approximately how long will it take for the deposit to triple in value?

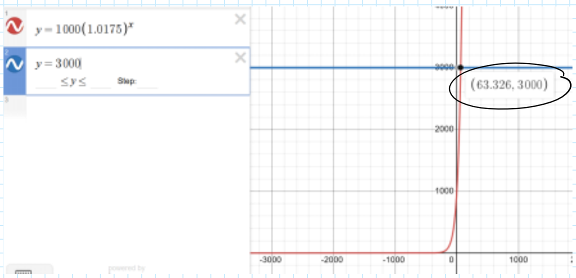
c) Does the amount of time it takes for a deposit to triple depend on the value of the initial deposit? Explain.









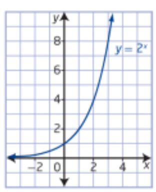


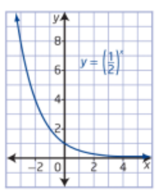
Invest your money wisely......

TB, p 342

Key Ideas

- An exponential function of the form $y = c^x$, $c > 0$,
  - is increasing for $C > 1$
  - is decreasing for $0 < c < 1$
  - is neither increasing nor decreasing for $C = 1$
  - has a domain of $x \in \mathbb{R}$
  - has a range of $y > 0$
  - has a y-intercept of $(0, 1)$
  - has no x-intercept
  - has a horizontal asymptote at $\longrightarrow$ $y = 0$





<u>Textbook Practice</u>
(7.1) p 342: 1, 3-8

7.2 Transformations of Exponential Functions

$y=5^x$

Predict what will happen to the graph of $y = 5^x$ when each of the following changes is made to the equation:

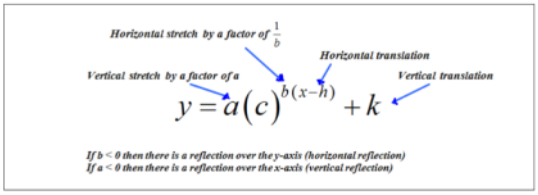
$y = 5^x + 1$ up 1

$y = 5^{x-4}$ right 4

$y = 5^{2x}$ H Compression by ½

$y = 5^{3x-12}$ $y=5^{3(x-4)}$ HC by 1/3, 4 right

$y = -2(5^x)$ Vertical expansion by −2 ~~OR~~ VE by 2, reflect across the x-axis

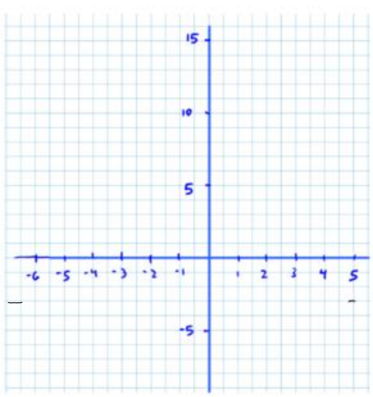


whiteboard
TB, p 351 - do in a group of 2 or 3

Transforming Exponential Graphs

Use the graph of $y = 4^x$ to create the graph of $y = 4^{-2(x+5)} - 3$.

1. Make a table of key points for the BASE function.
2. List all the transformations
3. Determine the mapping notation.
4. Make a table showing the final image points.
5. Draw in the horizontal asymptote, using a dotted line.
6. Plot the final image points, being careful not to cross the asymptote.
7. Give the domain, range, and horizontal asymptote equation for the final, transformed graph.



Base Function Table

Transformed Function Table

Transformations are:

Mapping is:

Domain:

Range:

Horizontal Asymptote Equation:

Transforming Exponential Graphs

Use the graph of $y = 4^x$ to create the graph of $y = 4^{-2(x+5)} - 3$.

1. Make a table of key points for the BASE function.
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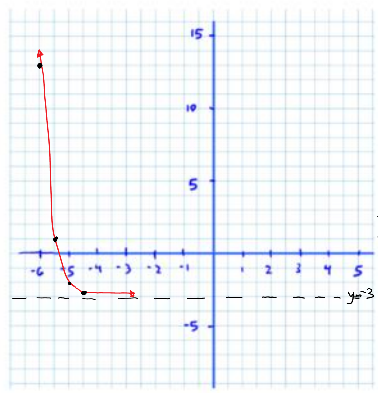


Base Function Table $y=4^x$

x	y
−1	$4^{-1} = \frac{1}{4}$
0	$4^0 = 1$
1	$4^1 = 4$
2	$4^2 = 16$

Transformed Function Table $y = 4^{-2(x+5)} - 3$

$-\frac{1}{2}x - 5$	$y - 3$
$-\frac{1}{2}(-1) - 5 = -4\frac{1}{2}$	$\frac{1}{4} - 3 = -1\frac{1}{4}$ or $-2\frac{3}{4}$
$-\frac{1}{2}(0) - 5 = -5$	$1 - 3 = -2$
$-\frac{1}{2}(1) - 5 = -5\frac{1}{2}$	$4 - 3 = 1$
$-\frac{1}{2}(2) - 5 = -6$	$16 - 3 = 13$

Transformations are: HC ½, reflect across y-axis, 5 left, 3 down

Mapping is: $(x, y) \rightarrow (-\frac{1}{2}x - 5, y - 3)$

Domain: $\{x \mid x \in \mathbb{R}\}$

Range: $\{y \mid y > -3, y \in \mathbb{R}\}$

Horizontal Asymptote Equation: $y = -3$

Back to notes package, page 3:

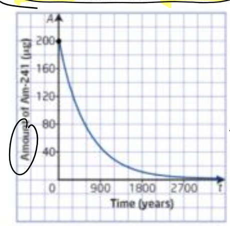
Creating Exponential Functions

–, it's

Back to notes package, page 3:

Creating Exponential Functions

The radioactive element americium is used in household smoke detectors. Am-241 has a half-life of approximately 432 years. The average smoke detector contains 200 μg of Am-241. What is the exponential function that models the graph showing the radioactive decay of 200 μg of Am-241? (TB, p 353)



– it's decay

time	Amount
0	200
432	100
864	50
1296	25

$y = \left(\frac{1}{2}\right)^x$

x	y
0	1
1	1/2
2	1/4

$A = 200\left(\frac{1}{2}\right)^{\frac{t}{432}}$

this corrects our y-values

this correct the amount of time needed to "halve"

$y = 200\left(\frac{1}{2}\right)^{\frac{x}{432}}$

$y = 200\left(\frac{1}{2}\right)^{\frac{1}{432}x}$

(7.2) TB p 354: 1, 2, 3adeg, 4, 5, 6cd, 7ac, 11a, 12a

****Create the Equation, notes page 4 -**
Do with a partner, using small whiteboards**

Pre-Calc 12 – Unit 3
Page 4

Create the Equation

1) The population of a town triples every 6 years. Suppose that 4000 people lived in the town in the year 2006.

a) equation

$A = 4000(3)^{t/6}$

b) How many people would be living in the town in 2050?

$A = 4000(3)^{(44/6)} \doteq$ 12 616,799 people (rounded down)

2) A bacterial culture doubles every 2 hours. This culture had 22 000 bacteria at time $t = 0$.

a) equation

$A = 22\,000(2)^{t/2}$

b) How many bacteria would be in the culture after 5 hours?

$A = 22\,000(2)^{(5/2)} =$ 124 450 bacteria

3) The half-life of a radioactive sample is 4 hours. The sample size was originally 60 g.

a) equation

$A = 60\left(\frac{1}{2}\right)^{t/4}$

b) How many grams would be in the sample after 11 hours?

$A = 60\left(\frac{1}{2}\right)^{11/4} =$ 8.9 grams

4) For every meter that you descend into water, 5% of light is blocked.
(If you start with 100% of the light and 5% is blocked, what percentage of light do you still have?) → 100% – 5% = 95%

a) equation

$A = 100(0.95)^{t/1}$ $\quad A = (0.95)^t$

b) What percentage of light would still pass through the water, at a depth of 15 meters?

$A = 100(0.95)^{15} = 46.329... \doteq 46\%$ $\quad A = (0.95)^{15} = 0.46329... \doteq 46\%$

5) $5000 is invested at 7.2% compounded **annually**.

7.2% = 0.072

a) equation

$A = 5000(1.072)^{t/1}$

b) How much money would you have after 3 years?

$A = 5000(1.072)^3 =$ $6159.62

Pre-Calc 12 – Unit 3
Page 4

amount of time for tripling to occur (horizontal stretch)

Create the Equation

Create the Equation

1) The population of a town triples every 6 years. Suppose that 4000 people lived in the town in the year 2006.

a) equation

$P = 4000(3)^{\frac{t}{6}}$

amount of time for tripling to occur (Horizontal stretch)

initial amount given / VE amount

b) How many people would be living in the town in 2050?

$P = 4000(3)^{\frac{44}{6}}$ time, after 2006 $\Rightarrow t = 44$ → $P = 12,616,799.24$ people

don't cut people into pieces!!

round down to nearest whole person

12,616,799 people

2) A bacterial culture doubles every 2 hours. This culture had 22 000 bacteria at time $t = 0$.

a) equation

$B = 22\,000(2)^{\frac{t}{2}}$

b) How many bacteria would be in the culture after 5 hours?

$B = 22\,000(2)^{\frac{5}{2}} = 124\,450.79$ round down to 124 450 bacteria

3) The half-life of a radioactive sample is 4 hours. The sample size was originally 60 g.

a) equation

$A = 60\left(\frac{1}{2}\right)^{\frac{t}{4}}$

b) How many grams would be in the sample after 11 hours?

$A = 60\left(\frac{1}{2}\right)^{\frac{11}{4}} \doteq 8.9$ g

4) For every meter that you descend into water, 5% of light is blocked.
(If you start with 100% of the light and 5% is blocked, what percentage of light do you still have?)

100% − 5% = 95% is what we have

a) equation

$A = (0.95)^{\frac{t}{1}}$

b) What percentage of light would still pass through the water, at a depth of 15 meters?

$A = (0.95)^{15} = 0.4632912302 \Rightarrow$ 46 %

5) \$5000 is invested at 7.2% compounded annually.

a) equation

$A = 5000(1 + 0.072)^{\frac{t}{1}}$, $A = 5000(1.072)^{\frac{t}{1}}$

b) How much money would you have after 3 years?

$A = 5000(1.072)^3$

$A = 6159.62624 \Rightarrow$ round down to nearest cent \$6159.62

In 2022, the population of Abbotsford, BC, was about 168,000. Its annual growth rate was 2.2%. $\Rightarrow$ 1.022 is our base

Create an equation that describes this situation and use it to estimate what the population will be in 2030 if the growth rate stays the same. (Round down, giving the answer correct to the nearest whole person.)

$A = 168{,}000(1.022)^{\frac{t}{1}}$

time = 2030 − 2022 = 8 years

$A = 168\,000(1.022)^8$

$\doteq 199,947$ people

Practice

(7.1) p 342: 1, 3-8

(7.2) TB p 354: 1, 2, 3adeg, 5, 6cd, 7ac, 11a, 12a