Class_17 Mar 7 - Coterminal Angles, Sine Law

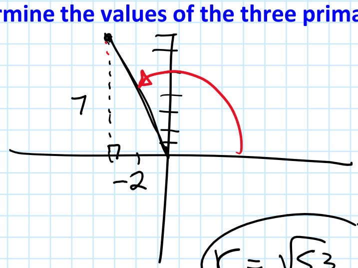
Wednesday, March 1, 2023 5:15 PM

Tonight's Class:

- **Unit 2 Test return and rewrite sign-up**
- **Warm-up**
- **Working through sections 5.2, 5.3, 5.5**
  - **Trig Ratios in All Quadrants (continued)**
  - **Coterminal Angles (5.3)**
  - **Sine Law (5.5)**

Warm-up

The point (-2, 7) is on the terminal arm of a standard position angle, θ.
Determine the values of the three primary trig ratios for θ.



$x^2 + y^2 = r^2$

$(-2)^2 + (7)^2 = r^2$

$4 + 49 = r^2$

$53 = r^2$, $r = \pm\sqrt{53}$

$r = \sqrt{53}$

$\sin\theta = \frac{y}{r} = \frac{7}{\sqrt{53}}$

$\cos\theta = \frac{x}{r} = \frac{-2}{\sqrt{53}}$

$\tan\theta = \frac{y}{x} = \frac{7}{-2}$

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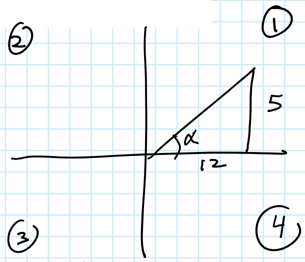
Example 4 **Determining Trigonometric Ratios Given Information about an Angle**

a) Given that $\tan\alpha = \frac{5}{12}$, determine the exact values of the other primary trigonometric ratios of the angle α. all of them!

b) To the nearest degree, determine the possible values for α when $0° \leq \alpha \leq 360°$.

$\tan\alpha = \frac{5}{12}$ ← y (opp), ← x (adj)



$x^2 + y^2 = r^2$

$(12)^2 + (5)^2 = r^2$

$144 + 25 = r^2$

$169 = r^2$

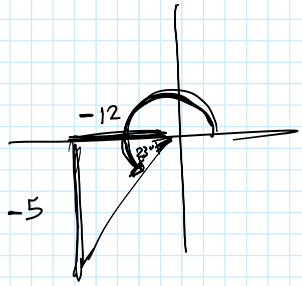
$\sqrt{169} = r$

$\boxed{r = 13}$

$\frac{5}{12}$ is positive.

Where is tangent positive?

Where is tangent positive?



Q1: $\sin\alpha = \frac{y}{r} = \frac{5}{13}$ $\cos\alpha = \frac{x}{r} = \frac{12}{13}$

$\tan\theta = \frac{-5}{-12}$

$= \frac{5}{12}$

Q3: $\sin\alpha = \frac{-5}{13}$ $\cos\alpha = \frac{-12}{13}$

b) What is α? $\tan\alpha = \frac{5}{12}$

$\alpha = \tan^{-1}\left(\frac{5}{12}\right) = \boxed{23^\circ}$ Q1 angle

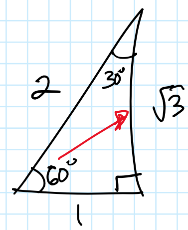
Q3 angle: $180^\circ + 23^\circ$

$= \boxed{203^\circ}$

Examples using special triangles

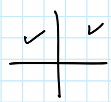
If $\sin\theta = \frac{1}{2}$, find the exact values of the other trigonometric ratios.

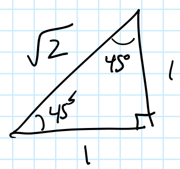
Find the values of θ that satisfy the equation, for $0^\circ \le \theta \le 360^\circ$.



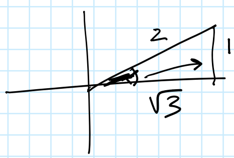
1) Where can this happen? $\sin\theta = +\frac{1}{2}$ ← y ← r

Q1, Q2



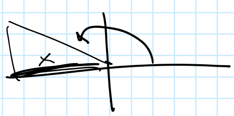


2)



$\cos\theta = \frac{x}{r} = \frac{\sqrt{3}}{2}$

$\tan\theta = \frac{y}{x} = \frac{1}{\sqrt{3}}$



$\cos\theta = -\frac{\sqrt{3}}{2}$

$\tan\theta = -\frac{1}{\sqrt{3}}$

3) Find angles:

$\sin\theta = \frac{1}{2}$

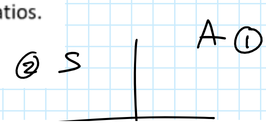
We can get θ, without having to use the calculator.

Q1: $\theta = 30^\circ$

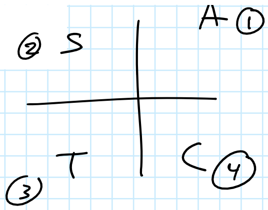
Q2: $180^\circ - 30^\circ = 150^\circ$

If $\cos\theta = -\frac{1}{2}$, find the exact values of the other trigonometric ratios.

Find the values of θ that satisfy the equation, for $0^\circ \le \theta \le 360^\circ$.



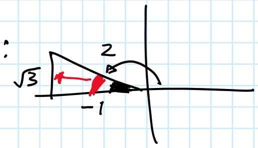
Find the values of θ that satisfy the equation, for $0^\circ \le \theta \le 360^\circ$.



1) $\cos\theta = -\frac{1}{2}$

$\Rightarrow$ Quadrants 2, 3

2) Quadrant 2:



$\sin\theta = \frac{y}{r}$

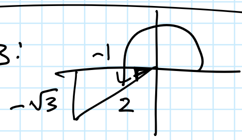
$= \frac{\sqrt{3}}{2}$

$\tan\theta = \frac{y}{x}$

$= \frac{\sqrt{3}}{-1}$

$= -\sqrt{3}$

Quadrant 3:



$\sin\theta = \frac{y}{r}$

$= \frac{-\sqrt{3}}{2}$

$\tan\theta = \frac{y}{x}$

$= \frac{-\sqrt{3}}{-1}$

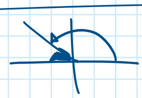
$= \sqrt{3}$

3) $\theta_R = 60^\circ$

$\cos\theta = -\frac{1}{2}$

$\theta_R = \cos^{-1}\left(+\frac{1}{2}\right) = 60^\circ$

Q_2



$\theta = 180^\circ - 60^\circ = \boxed{120^\circ}$

Q_3

$\theta = 180^\circ + 60^\circ = \boxed{240^\circ}$

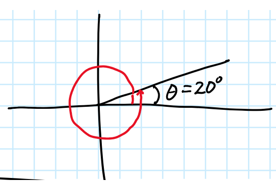
WT, page 433: #14hi, #15df, 16, 19

5.3 Coterminal Angles

Focus: Find trigonometric ratios for angles smaller than 0 degrees and greater than 360 degrees.

Coterminal Angles are
- different in size than the original angle but
- have the same terminal arm as the original angle

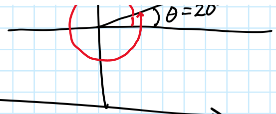
The collection of all angles coterminal to θ is given by: $\theta + 360^\circ n,\ n \in I$



380° is a coterminal angle

another one:

$20^\circ - 360^\circ = -340^\circ$



coterminal angle

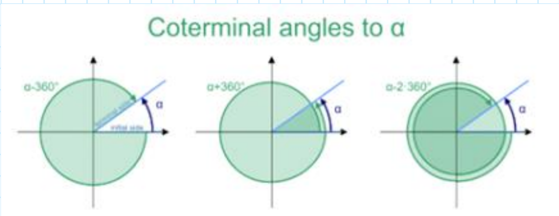
another one:

$20^\circ - 360^\circ = -340^\circ$

$20^\circ + 2(360^\circ) = 740^\circ$

all coterminal to 20°:

$$\boxed{20^\circ + 360^\circ n, \quad n \in I}$$



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Example 1 **Identifying and Sketching Coterminal Angles**

a) Determine the measures of all the angles in standard position from 0° to 1500° that are coterminal with an angle of 120° in standard position. Sketch the angles.

b) Write an expression for the measures of all angles that are coterminal with an angle of 120° in standard position.

120°

coterminal angles:

$120^\circ + 360^\circ = 480^\circ$

$120^\circ + 2(360^\circ) = 840^\circ$

$120^\circ + 3(360^\circ) = 1200^\circ$

$$\boxed{\theta = 120^\circ + 360^\circ n, \quad n \in I}$$

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the smallest POSITIVE angle that is coterminal to the given angle.

Example 2 **Determining the Principal Angle**

a) Determine the principal angle for −908°.

b) Use the principal angle to write an expression for the measures of all the angles that are coterminal with −520° in standard position.

a) -908°

$-908^\circ + 360^\circ = -548^\circ$

$-548^\circ + 360^\circ = -188^\circ$

$-188^\circ + 360^\circ = \boxed{172^\circ}$

this is the principal angle.

b) -520° First, let's find the principal angle:

$-520° + 360° = -160°$

$-160° + 360° = 200°$ ← principal angle

$$\boxed{200° + 360°n,\ n \in I}$$

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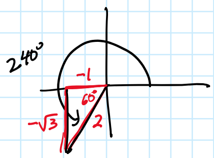
Example 3	**Determining Trigonometric Ratios of Angles Greater than 360° or Less than 0°**

a) Determine the exact values of the trigonometric ratios for −840°.
b) Determine the approximate values of the trigonometric ratios for −840°. Give the answers to 3 decimal places where necessary.

a) Let's find the principal angle for $-840°$:

$$-840° + 3(360°) = \boxed{240°}$$



We know the reference angle is 60°, and that helps us label the triangle sides

$\sin 240° = \frac{-\sqrt{3}}{2}$

$\cos 240° = \frac{-1}{2}$

$\tan 240° = \frac{-\sqrt{3}}{-1} = \frac{\sqrt{3}}{1} = \sqrt{3}$

b) approximate values, to 3 decimal places

$\sin(-840°) = -0.866$

$\cos(-840°) = -0.5$

$\tan(-840°) = 1.732$

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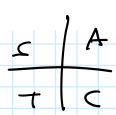
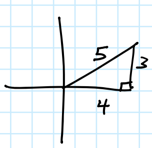
Example 4	**Determining All Trigonometric Ratios Given Information about an Angle**

a) Given that $\tan \alpha = \frac{3}{4}$, determine the exact values of the other primary trigonometric ratios of the angle α.
b) To the nearest degree, determine the possible values of α when $-360° \le \alpha \le 720°$.



a) $\tan \alpha = 3/4$

1) where? Q1, Q3



$\tan \alpha = \frac{3}{4}$ ← y, ← x

2) Q1: $\sin \alpha = \frac{3}{5}$

$\cos \alpha = \frac{4}{5}$

$x^2 + y^2 = r^2$

$3^2 + 4^2 = r^2$

$9 + 16 = r^2$

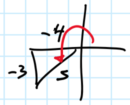
2) Q1: $\sin\alpha = \frac{3}{5}$

$\cos\alpha = \frac{4}{5}$

$x^2+y^2=r^2$
$3^2+4^2=r^2$
$9+16 = r^2$
$25 = r^2$, $r=\sqrt{25}$, $r=5$

Q3: $\sin\alpha = -\frac{3}{5}$

$\cos\alpha = -\frac{4}{5}$



3) $\tan\alpha = \frac{3}{4}$

start by getting one answer from the calculator:

$\tan^{-1}\left(\frac{3}{4}\right) \doteq \boxed{37^\circ}$ ← Q1 answer

Q3 answer: $180^\circ + 37^\circ = \boxed{217^\circ}$

4) Get all the coterminals to these two answers, between $-360^\circ \le \alpha \le 720^\circ$

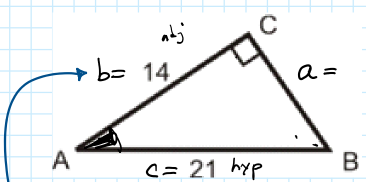
$37^\circ + 360^\circ = \boxed{397^\circ}$
$37^\circ + 2(360^\circ)$ = too big
$37^\circ - 360^\circ = \boxed{-323^\circ}$

$217^\circ + 360^\circ = \boxed{577^\circ}$
$217^\circ - 360^\circ = \boxed{-143^\circ}$

5.5 Sine Law

Focus: Apply the Sine Law to solve problems in triangles that are not right triangles.

Solve this triangle
- find all the side lengths
- find all the angles



remember we label the side opposite an angle with the same letter.

We know: $a^2 + 14^2 = 21^2$

$a^2 = 21^2 - 14^2$

$a^2 = 245$

$a = \sqrt{245}$

$a = \sqrt{49 \cdot 5}$

$\boxed{a = 7\sqrt{5}}$ ← exact form

approximate answer: $a \doteq 15.7$, to one decimal place

$\cos A = \frac{adj}{hyp}$

$\cos A = \frac{14}{21}$

$A = \cos^{-1}\left(\frac{14}{21}\right)$

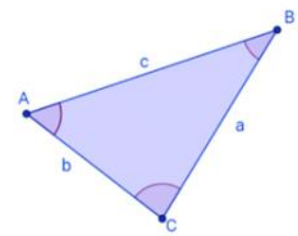
$\boxed{A \doteq 48.2^\circ}$

$B = 180^\circ - 90^\circ - 48.2^\circ$

$\boxed{B \doteq 41.8^\circ}$

The following figure shows the Law of Sines for the triangle ABC

True for any triangle.



Law of sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

We can also write the law of sines or sine rule as:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

To use the Sine Law we need to be given

- **two angles and one side of the triangle (AAS or ASA)**

OR

- **two sides and an angle opposite one of them (SSA)**

we must have one matching pair

angle and a side that are across from each other

Two Angles and One Side

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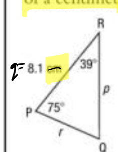
Example 1 **Using the Sine Law to Determine the Length of a Side**

In ΔPQR, determine the length of QR to the nearest tenth of a centimetre.



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

write one of the things you need to find out on the top

$P = 75^\circ$

$R = 39^\circ$

$Q = 180^\circ - 75^\circ - 39^\circ$

$\boxed{Q = 66^\circ \quad q = 8.1}$

$$\frac{p}{\sin P} = \frac{q}{\sin Q}$$

$$\sin 75^\circ \cdot \left(\frac{p}{\sin 75^\circ}\right) = \frac{8.1}{\sin 66^\circ} \cdot \sin 75^\circ$$

$$p = \frac{8.1 \sin 75^\circ}{\sin 66^\circ}$$

$$\boxed{p \doteq 8.6 \text{ cm}}$$

$$\frac{r}{\sin R} = \frac{q}{\sin Q}$$

$$\sin 39^\circ \left(\frac{r}{\sin 39^\circ}\right) = \left(\frac{8.1}{\sin 66^\circ}\right) \sin 39^\circ$$

$$r = \frac{8.1 \sin 39^\circ}{\sin 66^\circ}$$

$$\boxed{r \doteq 5.6 \text{ cm}}$$

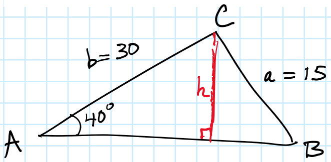
p469 CYU #1 ; p475 #3

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Two Sides and One Angle

Example

Solve ΔABC, given that $A = 40^\circ$, $a = 15\,cm$, $b = 30\ cm$



(we don't try to draw it accurately)

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\cancel{30}\left(\frac{\sin B}{\cancel{30}}\right) = \left(\frac{\sin 40^\circ}{15}\right) 30$$

$$\sin B = \frac{30 \sin 40^\circ}{15}$$

$$B = \sin^{-1}\left(\frac{30 \sin 40^\circ}{15}\right)$$

$B =$ error ?! Why?

15 cm won't reach!!

$\Rightarrow \boxed{\text{no } \Delta}$

$$30(\sin 40^\circ) = \frac{h}{\cancel{30}} \cdot \cancel{30}$$

$$30(\sin 40^\circ) = h$$

$$19.3 = h$$

This is the shortest possible length.

When we're given SSA...

AMBIGUOUS

https://youtu.be/BMr4m5AEqlQ?t=69 (time-stamped)

$a < h$	No possible triangle
$a = h$	One possible right-angle triangle
$h < a < b$	The ambiguous case! There are 2 possible triangles
$a \geq b$	One possible triangle

$a \leq b$	No possible triangle
$a > b$	One possible triangle

[Determining if Triangles Exist - Nerdstudy](#)

[The Ambiguous Case for Sine Law - Nerdstudy](#)

[How to Solve the Ambiguous Case WITHOUT MEMORIZING Anything (with Visuals)](#)

https://youtu.be/BMr4m5AEqlQ?t=69

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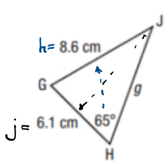
Solve ΔGHJ, shown in the diagram.



$$\frac{\sin J}{j} = \frac{\sin H}{h}$$

$$\cancel{6.1}\left(\frac{\sin J}{\cancel{6.1}}\right) = \left(\frac{\sin 65°}{8.6}\right) \cdot 6.1$$

$$\sin J = \frac{6.1 \sin 65°}{8.6}$$

Is there another triangle?

$$\boxed{H = 65°}$$

H = 65°
h = 8.6 cm

$H = 65^\circ$
$h = 8.6\text{ cm}$
$j = 6.1\text{ cm}$

$$\sin J = \frac{6.1 \sin 65^\circ}{8.6}$$

$$J = \sin^{-1}\left(\frac{6.1 \sin 65^\circ}{8.6}\right)$$

$$\boxed{J = 40^\circ}$$

$$G = 180^\circ - 65^\circ - 40^\circ$$

$$\boxed{G = 75^\circ}$$

Solve for g:

$$\frac{g}{\sin G} = \frac{h}{\sin H}$$

$$\cancel{\sin 75^\circ}\left(\frac{g}{\cancel{\sin 75^\circ}}\right) = \left(\frac{8.6}{\sin 65^\circ}\right)\sin 75^\circ$$

$$g = \frac{8.6 \sin 75^\circ}{\sin 65^\circ}$$

$$\boxed{g \doteq 9.2\text{ cm}}$$

$$J = 180^\circ - 40^\circ$$

$$\boxed{J = 140^\circ}$$

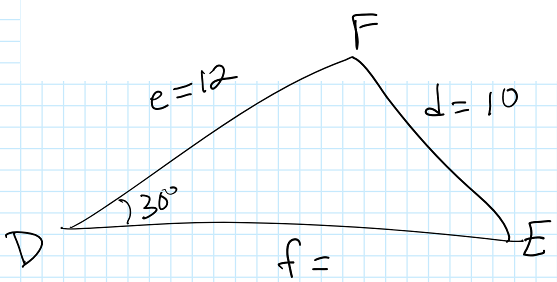
Is this possible?

$$\begin{array}{r} 65^\circ \\ +140^\circ \\ \hline 205^\circ \end{array}$$

Already more than 180°.

$\Longrightarrow$ This triangle can't be created! There's only 1 triangle.

Example
Solve ΔDEF, given that $D = 30^\circ$, $d = 10$, $e = 12$



$$\frac{\sin E}{e} = \frac{\sin D}{d}$$

$$\cancel{12}\left(\frac{\sin E}{\cancel{12}}\right) = \left(\frac{\sin 30^\circ}{10}\right)\cdot 12$$

$$\sin E = \frac{12 \sin 30^\circ}{10}$$

$$E = \sin^{-1}\left(\frac{12 \sin 30^\circ}{10}\right)$$

$$\boxed{E \doteq 37^\circ}$$

$$F = 180^\circ - 30^\circ - 37^\circ$$

Is there another triangle?

$$E = 180^\circ - 37^\circ$$

$$\boxed{E = 143^\circ}$$

$$D = 30^\circ$$

$$F = 180^\circ - 143^\circ - 30^\circ$$

$$F = 180° - 30° - 37°$$

$$\boxed{F = 113°}$$

$$\frac{f}{\sin F} = \frac{d}{\sin D}$$

$$\cancel{\sin 113°}\left(\frac{f}{\cancel{\sin 113°}}\right) = \left(\frac{10}{\sin 30°}\right)\sin 113°$$

$$f = \frac{10 \sin 113°}{\sin 30°} \doteq \boxed{18.4}$$

$$D = 30$$

$$F = 180° - 143° - 30°$$

$$\boxed{F = 7°}$$

$$\frac{f}{\sin 7°} = \frac{10}{\sin 30°}$$

$$f = \frac{10 \sin 7°}{\sin 30°}$$

$$\boxed{f \doteq 2.4}$$

IN SUMMARY,

1. Find the second angle of the triangle. If it doesn't make sense, then there's no triangle.

2. See if the supplementary angle of that angle can also make a triangle. If so, then there are two triangles. If not, then there's just one triangle.

3. Solve for the rest of the triangle(s) using Law of Sines

For next class

- **Work on the worktext questions for chapter 5**
- **We are omitting section 5.4**
- **I'll have the Chapter 5 hand-in assignment for you, next class. (It will be due March 28, when we return from spring break.)**