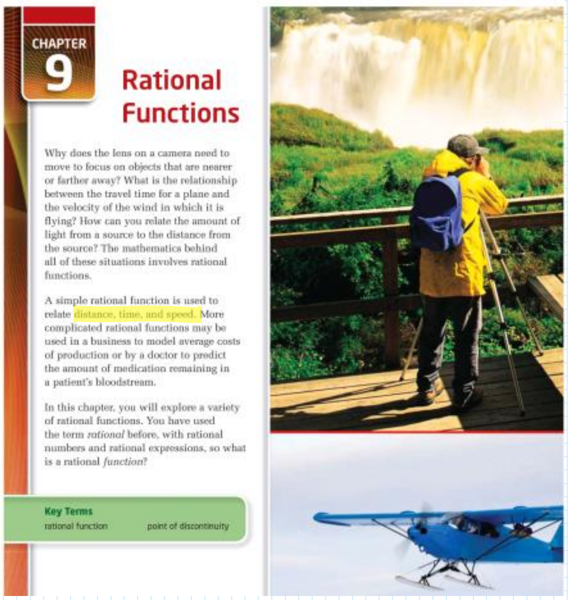
Class_21 June 8 -Rational Functions

Wednesday, June 7, 2023 11:45 AM

Tonight's Class:
- **9.1 Exploring Rational Functions**
- **9.2 Analyzing Rational Functions**

Coming up
- **Monday, June 12**
  - **Test 6 (8.2-8.4, 9.1-9.2)**
  - **Chapter 8 Hand-in due**
- **Thursday, June 15**
  - **Chapter 9 Hand-in due**
- **Tuesday, June 20**
  - **Test 7 (9.3, G.1-G.4)**
  - **Chapter G (10) Hand-in due**
- **Wednesday, June 21**
  - **Rewrite day (optional, can do up to 2 test rewrites)**



CHAPTER 9

Rational Functions

Why does the lens on a camera need to move to focus on objects that are nearer or farther away? What is the relationship between the travel time for a plane and the velocity of the wind in which it is flying? How can you relate the amount of light from a source to the distance from the source? The mathematics behind all of these situations involves rational functions.

A simple rational function is used to relate distance, time, and speed. More complicated rational functions may be used in a business to model average costs of production or by a doctor to predict the amount of medication remaining in a patient's bloodstream.

In this chapter, you will explore a variety of rational functions. You have used the term *rational* before, with rational numbers and rational expressions, so what is a rational *function*?

Key Terms

rational function point of discontinuity

Chapter 9: Rational Functions

9.1 Exploring Rational Functions Using Transformations

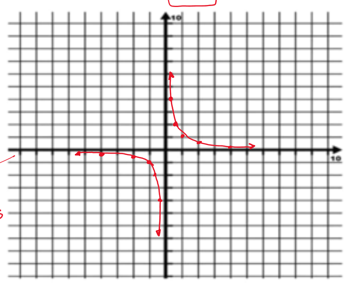
Rational function are functions that can be written in the form $y = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.

Rational Functions with Linear Numerators and Denominators

Base rational function, $\boxed{y = \frac{1}{x}}$

two "branches" of the function

not continuous



$y = \frac{1}{x}$	
x	y
-4	$\frac{1}{-4}$
-2	$\frac{1}{-2}$
-1	$\frac{1}{-1} = -1$
$-\frac{1}{2}$	$\frac{1}{-1/2} = 1 \cdot \frac{-2}{1} = -2$
$-\frac{1}{4}$	$\frac{1}{-1/4} = -4$
0	$\frac{1}{0}$ = undefined
$\frac{1}{4}$	4
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
4	$\frac{1}{4}$

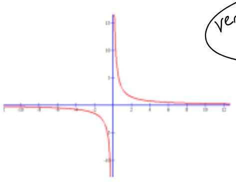
$\boxed{y = \frac{1}{x}}$

Non-permissible value (NPV) any x-value that makes denominator equal 0	$x = 0$
End behavior As $\lvert x \rvert$ becomes very large, what does y do?	y approaches zero $y \to 0$
Domain	$\{x \mid x \neq 0, x \in \mathbb{R}\}$
Range	$\{y \mid y \neq 0, y \in \mathbb{R}\}$
Equation of vertical asymptote	$x = 0$
Equation of horizontal asymptote	$y = 0$

In general, these are the ***simplest*** rational function equations, together with their graphs:

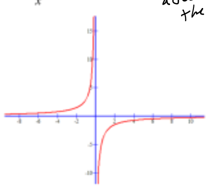
$y = \frac{a}{x}$, $a > 0$ $y = a\left(\frac{1}{x}\right)$

vertical stretch by a



$y = \frac{a}{x}$, $a < 0$

reflects across the x-axis.



Transformations of $y = \frac{1}{x}$

For $y = \frac{a}{x-h} + k$, complete the table below.

vertical translation

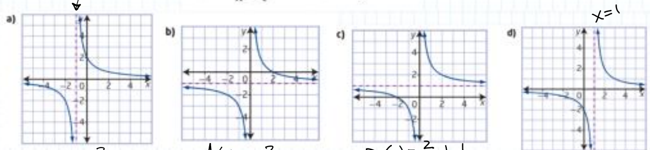
horizontal translation

Non-permissible value (NPV) * look at the denominator	$x = h$
End behavior As $\lvert x \rvert$ becomes very large, what does y do?	y approaching k
Domain	$\{x \mid x \neq h, x \in \mathbb{R}\}$
Range	$\{y \mid y \neq k, y \in \mathbb{R}\}$
Equation of vertical asymptote	$x = h$
Equation of horizontal asymptote	$y = k$

TB p 442: 1

$A(x) = \frac{2}{x} - 1 \qquad B(x) = \frac{2}{x+1}$

$C(x) = \frac{2}{x-1} \qquad D(x) = \frac{2}{x} + 1$



$B(x) = \frac{2}{x+1}$ $\quad A(x) = \frac{2}{x} - 1$ $\quad D(x) = \frac{2}{x} + 1$ $\quad C(x) = \frac{2}{x-1}$

To Try

1. Given the original rational function $y = \frac{1}{x}$ and the transformed function, $y = \frac{4}{x+3} - 5$:

a) Complete the tables below. For the first table, give 6 points found on the graph of the ***original function*** $y = \frac{1}{x}$. In the final table, give the image points that result after the transformations have occurred. Write the mapping notation in the heading of that table.

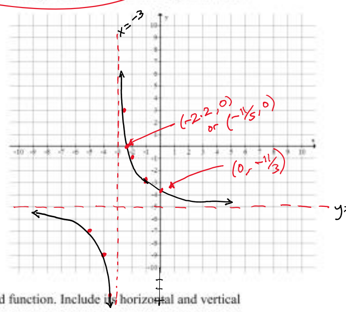
$(x, y) \rightarrow (x-3, 4y-5)$

$y = 4\left(\frac{1}{x+3}\right) - 5$

$y = \frac{1}{x}$ $\qquad \boxed{y = \frac{4}{x+3} - 5}$

x	y
-2	-1/2
-1	-1
-1/2	-2
1/2	2
1	1
2	1/2

x-3	4y-5	
-5	-7	$4(-\frac{1}{2}) - 5$
-4	-9	$4(-1) - 5$
$-3\frac{1}{2}$	-13	$4(-2) - 5$
$-2\frac{1}{2}$	3	$4(2) - 5$
-2	-1	$4(1) - 5$
-1	-3	$4(\frac{1}{2}) - 5$



b) Accurately sketch the final transformed function. Include the horizontal and vertical asymptotes, drawn with dotted lines.

c) Give the equations of the asymptotes.

horizontal asymptote $y = -5$ $\qquad$ vertical asymptote $x = -3$

d) Use algebra to find the coordinates of the final graph's x-intercept and y-intercept.

x-intercept, let $y = 0$

$y = \frac{4}{x+3} - 5$

$0 = \frac{4}{x+3} - 5$

$(x+3)5 = \frac{4}{\cancel{x+3}}\cancel{(x+3)}$

$5(x+3) = 4$

$5x + 15 = 4$

$\frac{5x}{5} = \frac{-11}{5}$

$x = \frac{-11}{5}$

$\left(\frac{-11}{5}, 0\right)$

y-intercept, let $x = 0$

$y = \frac{4}{x+3} - 5$

$y = \frac{4}{0+3} - 5$

$y = \frac{4}{3} - \frac{5}{1} \cdot \frac{3}{3}$

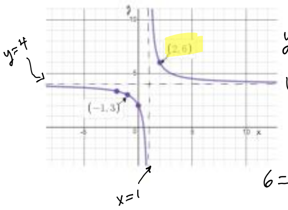
$y = \frac{4}{3} - \frac{15}{3}$

$y = -11/3$

$\left(0, -\frac{11}{3}\right)$

It will be in this format $\longrightarrow$ $\boxed{y = \frac{a}{x-h} + k}$

2. Find the equation of the rational function graphed below.



$y = \frac{a}{x-1} + 4$

Use (2,6) in the equation:

$6 = \frac{a}{2-1} + 4$

$6 = \frac{a}{1} + 4$

$2 = a$

1) use the asymptotes from the graph to figure out "h" and "k"

2) see if there's a stretch, by putting "a" into your equation and replacing x and y with the values from a point on the graph. Solve, to find the a-value.

$\boxed{y = \frac{2}{x-1} + 4}$

Example

Given the rational function $y = \frac{-4x+3}{x+2}$

Find its NPVs, intercepts, and asymptote equations.

1) NPVs denom = 0

$x + 2 = 0$

$\boxed{x = -2}$

2) x-int, let y = 0 $(x+2)(0) = \left(\frac{-4x+3}{\cancel{x+2}}\right)\cancel{(x+2)}$

$0 = -4x + 3$

$\frac{4x}{4} = \frac{3}{4}$

$x = 3/4$

$\left(\frac{3}{4}, 0\right)$

(x-intercept is found where the numerator equals 0)

y-int, let x = 0 $y = \frac{-4x+3}{x+2}$

$y = \frac{-4(0)+3}{0+2}$

$y = 3/2$

$\left(0, \frac{3}{2}\right)$

3) vertical asymptote denom = 0

$x + 2 = 0$

$\boxed{x = -2}$

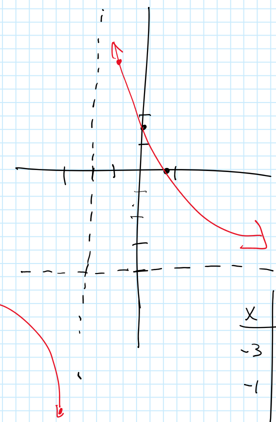
horizontal asymptote as x-values get HUGE, what are the y-values approaching?

$y = \frac{-4x+3}{x+2}$ try plugging in x = BIG number

$y = \frac{-4(\text{billion})+3}{(\text{billion})+2}$

$y \doteq -4$

y = -4 is the equation of horizontal asymptote



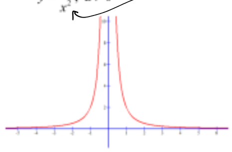
x	y
-3	-15
-1	7

$\frac{3}{-1} = \frac{15}{-1}$

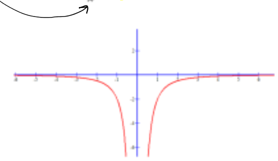
$= -15$

Rational Functions with Quadratic Term in Denominator, no other x's

$y = \frac{a}{x^2},\ a > 0$



$y = \frac{a}{x^2},\ a < 0$



Non-permissible value (NPV)	$x = 0$
Domain	$\{x \mid x \neq 0,\ x \in \mathbb{R}\}$
Range	$a > 0$ $\{y \mid y > 0,\ y \in \mathbb{R}\}$; $a < 0$ $\{y \mid y < 0,\ y \in \mathbb{R}\}$
Equation of vertical asymptote	$x = 0$
Equation of horizontal asymptote	$y = 0$

Transformations of $y = \frac{1}{x^2}$

For $\boxed{y = \frac{a}{(x-h)^2} + k}$, complete the table below.

Non-permissible value (NPV)	$x = h$
Domain	$\{x \mid x \neq h,\ x \in \mathbb{R}\}$
Range	$a > 0$ $\{y \mid y > k,\ y \in \mathbb{R}\}$; $a < 0$ $\{y \mid y < k,\ y \in \mathbb{R}\}$
Equation of vertical asymptote	$x = h$
Equation of horizontal asymptote	$y = k$

Example (TB p 445, #18)

Two stores rent bikes. The first store charges a fixed fee of \$20 plus \$4/h, and the second store charges a fixed fee of \$10 plus \$5/h.

a) Write equations for the average cost per hour for each store as a function of the rental time in hours.

Let x = # of hours the bike is rented

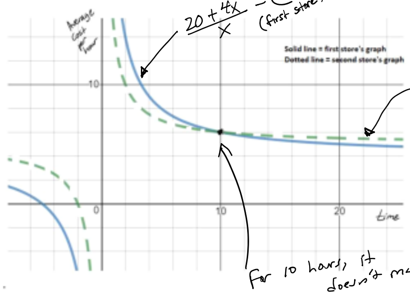
Store 1 : Average cost per hour $= \dfrac{20 + 4x}{x}$

Store 2 : Avg cost per hour $= \dfrac{10 + 5x}{x}$

b) Shown below are the graphs of the two equations from part (a). Which store is the better choice? Explain.

Depends

$\dfrac{20+4x}{x} = C$ (first store)



$\dfrac{10+5x}{x} = C$ (second store)

For less than 10 hours, the second store is less expensive.

For more than 10 hours, the first store is less expensive

For 10 hours, it doesn't matter.

Whiteboards - rational functions so far ("simplify" questions only)

9.2 Analyzing Rational Functions

Some rational function equations are more complicated. To analyze and graph them, we ***factor and simplify*** their equations.

Example

Consider the rational function: $f(x) = \dfrac{x^2+7x+12}{x+4}$

a) Factor and simplify the function's equation.

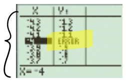
$f(x) = \dfrac{(x+3)\cancel{(x+4)}}{\cancel{x+4}}$

include this!

$\boxed{f(x) = x+3, \text{ where } x \neq -4}$

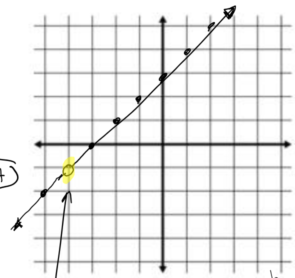
b) NPV(non-permissible value) = -4
How does the graph behave near its NPV?

x-values that are close to the NPV



$y = x+3$

x	$f(x)$
-5	-2
-4	Does not exist
-3	0
-2	1
-1	2
0	3
1	4
2	5



this hole shows the graph does not exist when $x = -4$

Point of Discontinuity (POD) – an ordered pair where the graph of a function does not exist. It occurs whenever the equation's numerator and denominator have a common factor that includes a variable.

POD

Example

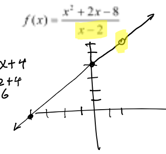
a) Complete the table, with the characteristics of the two graphs.

	$f(x) = \dfrac{x^2+2x-8}{x-2}$	$g(x) = \dfrac{x^2+6x+8}{x-2}$
Non-permissible value(s)	$x = 2$	$x = 2$
Simplified form of equation	$f(x) = \dfrac{\cancel{(x-2)}(x+4)}{\cancel{x-2}}$ $f(x) = x+4$, where $x \neq 2$	$g(x) = \dfrac{(x+2)(x+4)}{x-2}$ no simplifying can happen. There is NO POD.
Coordinates of x- and y-intercepts Can use the simplified version of the equation	x-int: $y = x+4$, $0 = x+4$, $x = -4$, $(-4, 0)$ y-int: $y = x+4$, $y = 0+4$, $y = 4$, $(0, 4)$	x-int: $(x-2)\,0 = \dfrac{(x+2)(x+4)}{x-2}\,(x-2)$, $0 = (x+2)(x+4)$, $x=-2$, $x=-4$, $(-2,0)$ and $(-4,0)$ y-int: $y = \dfrac{(0+2)(0+4)}{0-2}$, $y = \dfrac{(2)(4)}{-2}$, $y = \dfrac{8}{-2}$, $y = -4$, $(0,-4)$

b) Graph these rational functions (same as the ones above) using technology. Below each equation draw a rough sketch of its graph.

$f(x) = \dfrac{x^2+2x-8}{x-2}$

$f(x) = x+4$
$2+4$
$= 6$

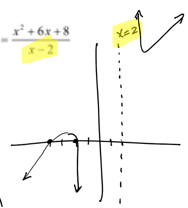


x	y
-4	0
0	4
2	DNE (6)

To figure out the y-value of the POD, we plug in $x=2$ to the simple equation.

$g(x) = \dfrac{x^2+6x+8}{x-2}$

$x=2$



When does a rational function have
- ***a point of discontinuity*** → if a factor containing x exists on both numerator & denominator, that are exactly the same
- ***a vertical asymptote?*** → if there's an NPV when the factor does NOT cancel

Whiteboards - PODs and horizontal asymptotes

Key Ideas for Rational Function Graphs

1) Horizontal Asymptotes

Find the degree of the numerator and denominator.

	Example	
Numerator degree < Denominator degree ***horizontal asymptote equation:*** $y = 0$	$y=\frac{2x}{3x^2+4}$	h.a. $y=0$
Numerator degree = Denominator degree ***horizontal asymptote equation:*** $y = \frac{\text{leading coefficient of num.}}{\text{leading coefficient of denom.}}$	$y=\frac{5x^2-7}{2x^2+3}$	h.a. $y=\frac{5}{2}$
Numerator degree > Denominator degree Graph will have a ***slant asymptote***	$y=\frac{5x^3}{2x}$	slant asymptote

$y=\frac{5}{x}+3$ shifted up

h.a. $y=$

2) NPVs, PODs, and vertical asymptotes

Factor numerator and denominator completely.

- Set ***each factor of the denominator = 0***, to get all NPVs.
  - Is there a factor that cancels with a factor in the numerator? It gives the x-value of a POD.
  - Is there a factor that doesn't cancel with a numerator factor? It gives the location of a vertical asymptote.

3) Intercepts

- ***y*-intercepts** – substitute $x = 0$ into the function (either the original or the simplified form) and solve for y
- ***x*-intercepts** - set each factor of the simplified numerator = 0 and solve for x

4) Sketch

- Plot all x-intercepts and y-intercepts
- Show points of discontinuity (PODs) as "holes", using an open circle
- Show all asymptotes as dotted lines.
- Find more points on the graph, as needed, by substituting into its equation.
- Make sure graph does not cross any vertical asymptotes.

<u>For next class</u>

- **Complete Chapter 8 Hand-in**
- **Complete Chapter 9 Hand-in, #1-3**
- **Prepare for Week 6 Test**

- **See optional worksheets on website, in both the Unit 3 and Unit 4 sections.**
- **More log scale questions:**
  - **TB p 401: 13bc, 16bc**
  - **TB p 417: 15, 17**
  - **TB p 419: 6, 15**