Class_21 Nov 22 - Log Applications, Rational Functions

Thursday, November 17, 2022 9:25 PM

Tonight's Class:
- **8.4 Log Equations and Applications (continued)**
- **9.1 Exploring Rational Functions**
- **Unit 3 Test (Chapters 7 and 8) on Thursday, Nov 24**

Here's a small strategy that makes a big impact on student learning – based on decades of cognitive science research. In scientific lingo, we call it **"free recall."**

Here's how it works:

1. Pause your lesson, lecture, or activity.
2. Ask students to write down everything they can remember.
3. Continue your lesson, lecture, or activity.

① $\log_3(2x+1) - \log_3(x) = 4$ $\quad \frac{1}{79}$

$\log_3\left(\frac{2x+1}{x}\right) = 4$

$3^4 = \frac{2x+1}{x}$

$(x)81 = \left(\frac{2x+1}{\cancel{x}}\right)\cancel{(x)}$

$81x = 2x+1$

$-2x \quad -2x$

$79x = 1$

$x = \frac{1}{79}$

② $5^{2x+1} = 7^{3x}$

$\log 5^{2x+1} = \log 7^{3x}$

$(2x+1)\log 5 = 3x\log 7$

$2x\log 5 + 1\log 5 = 3x\log 7$

$-2x\log 5 \quad -2x\log 5$

$\log 5 = 3x\log 7 - 2x\log 5$

$\log 5 = x(3\log 7 - 2\log 5)$

$\frac{\log 5}{(3\log 7 - 2\log 5)} = x$

Earthquakes, Sound, pH

Logarithms can be used to solve applications comparing the intensity of earthquakes, the intensity of sounds, and the acidity or alkalinity of solutions. The Richter scale for earthquakes, the decibel scale for sounds and the pH scale for solutions are all base 10.

$$I = I_0 (10)^{R-r}$$

where:
I = intensity of a stronger earthquake
I_o = intensity of weaker earthquake

R = Richter magnitude of stronger earthquake
r = Richter magnitude of weaker earthquake

$$I = I_0 (10)^{(D-d)/10}$$

where:
I = intensity of a louder sound
I_o = intensity of a softer sound

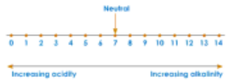
D = decibel level of louder sound
d = decibel level of softer sound

$$I = I_0 (10)^{P-p}$$

where:
I = new solution that is compared to original one
I_o = "original" solution

P = larger pH reading
p = smaller pH reading

- A neutral solution has a pH of 7.
- Solutions with pH larger than 7 are basic, or alkaline.
- Solutions with pH smaller than 7 are acidic.



$pH = -\log\left[H^+\right]$ where $\left[H^+\right]$ is the hydrogen ion concentration in moles per liter

The exponent is always a difference: larger reading – smaller reading
For sound questions, divide each decibel reading by 10.

$I = I_o(10)^{R-r}$ | $I = I_o(10)^{\frac{D-d}{10}}$ | $I = I_o(10)^{P-p}$

Example

1a) In 1983, an earthquake measuring 5.5 on the Richter scale occurred in Columbia. In 1989, the San Francisco earthquake measured 6.9 on the Richter scale. How much more intense was the San Francisco earthquake than the Columbia earthquake?

r=

R

$I = I_o(10)^{6.9-5.5}$

$I = I_o(10)^{1.4}$

strong earthquake is

$I \doteq I_o(25.12)$

25.12 × weak earthquake

25.12 times as intense

b) Calculate the magnitude of an earthquake that is 1500 times as intense as the Columbia earthquake.

5.5

$\frac{1500\cancel{I_o}}{\cancel{I_o}} = \frac{\cancel{I_o}(10)^{R-5.5}}{\cancel{I_o}}$

$1500 = 10^{R-5.5}$

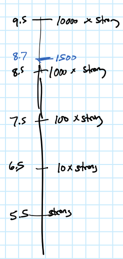
$\log(1500) = {}^{R-5.5}\log(10)$

$\log 1500 = (R-5.5)\cancel{\log_{10} 10}^{=1}$

$\log 1500 = R-5.5$

$\log(1500)+5.5 = R$

$R = 8.7$ on Richter scale



2. How much louder is a sound with an intensity of 112 dB compared to a sound with an intensity of 90 dB

$I = I_o(10)^{\frac{D-d}{10}}$

$I = I_o(10)^{\frac{112-90}{10}}$

$I = I_o(10)^{2.2}$

$I \doteq I_o(158.5)$

158.5 times as loud

b) If three different jets are flying together at an air show, each with a sound level of 120 decibels, then find the approximate total decibel level.

$I = I_o(10)^{\frac{D-d}{10}}$

$\frac{3\cancel{I_o}}{\cancel{I_o}} = \frac{\cancel{I_o}(10)^{\frac{D-120}{10}}}{\cancel{I_o}}$

$3 = 10^{\frac{D-120}{10}}$

$\log 3 = {}^{\frac{D-120}{10}}\log 10$

$\log 3 = \left(\frac{D-120}{10}\right)\cancel{\log 10}^{=1}$

$10 \times \log 3 = \frac{D-120}{\cancel{10}} \times \cancel{10}$

$10\log 3 \;{}_{+120} = D - 120 \;{}_{+120}$

$10\log(3)+120 = D$

$\boxed{D \doteq 124.8 \text{ dB}}$

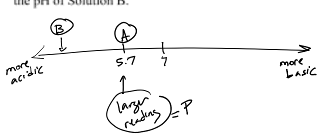
Example $\quad pH = -\log\left[H^+\right] \quad I = I_0(10)^{P-p}$

a) A beaker of acid has a hydrogen ion concentration of 3.5×10^{-6} mol/L. Calculate the pH of the acid.

$pH = -\log\left[3.5\times10^{-6}\right]$

$\doteq 5.5$ on pH scale

b) Solution A has a pH of 5.7. Solution B is 1260 times more **acidic** than Solution A. Find the pH of Solution B.



$I = I_o(10)^{P-p}$

$\frac{1260\cancel{I_o}}{\cancel{I_o}} = \frac{\cancel{I_o}(10)^{5.7-p}}{\cancel{I_o}}$

$1260 = 10^{5.7-p}$

$\log 1260 = {}^{5.7-p}\log 10$

$\log 1260 = (5.7-p)\cancel{\log 10}^{=1}$

$\log 1260 = 5.7 - p$
$\quad -5.7 \quad -5.7$

$\log(1260)-5.7 = -p$

$p \doteq 2.6$

$$\log 1260 = 5.7 - p$$
$$-5.7 \qquad -5.7$$
$$\log(1260) - 5.7 = -p$$
$$\frac{-2.599\ldots}{-1} = \frac{-p}{-1}$$
$$\boxed{p \doteq 2.6 \text{ on pH}}$$

For you to try . . .

1. How many times more intense is an earthquake with magnitude 8.3 than one with magnitude 6.7? (Round to nearest whole number.)

$$I = I_0(10)^{8.3-6.7}$$
$$I = I_0(10)^{1.6}$$
$$I \doteq I_0(39.81)$$
$$\boxed{\text{about } 40 \text{ times as intense}}$$

2. Bob was in an earthquake of magnitude 7.1. This earthquake was 420 times more intense than a smaller earthquake that his friend Joan was in. Find the magnitude of the smaller earthquake, correct to one decimal place.

$$420\cancel{I_0} = \cancel{I_0}(10)^{7.1-r}$$
$$420 = 10^{7.1-r}$$
$$\log 420 = \log 10^{7.1-r}$$
$$\log 420 = (7.1-r)\underbrace{\log 10}_{=1}$$
$$\log 420 = 7.1 - r$$
$$r = 7.1 - \log 420$$
$$\boxed{r \doteq 4.5 \text{ on Richter}}$$

3. How many times more intense is the sound of a power saw, 120 dB, than that of a leaf rustling, 10 dB?

$$I = I_0(10)^{\frac{120-10}{10}}$$
$$I = I_0(10)^{11}$$
$$\boxed{10^{11} \text{ times as intense}}$$

4. Two telephones in a home ring at the same time with a loudness of 80 decibels each. What is the decibel rating of the total loudness? (Note that 150d B is the sound of a jet engine, from 20 meters away, so the correct answer to this question is NOT 160 dB.)

$$2\cancel{I_0} = \cancel{I_0}(10)^{\frac{D-80}{10}}$$
$$\log 2 = \log 10^{\frac{D-80}{10}}$$
$$\log 2 = \left(\frac{D-80}{10}\right)\underbrace{\log 10}_{=1}$$
$$\log 2 = \frac{D-80}{10}$$
$$10\log 2 = D - 80$$
$$10\log(2) + 80 = D$$
$$\boxed{D \doteq 83\text{ dB}}$$

5. Determine the pH of a solution, to the nearest tenth, if they hydrogen ion concentration is 3.4×10^{-4} mol/L.

$$pH = -\log\left[3.4\times10^{-4}\right]$$
$$\doteq 3.5$$

6. Swimming pool water has a pH of 7.5. Sea water is about 8 times as alkaline as swimming pool water. What is the pH reading for sea water?

Pool 7.5 — Sea water — Alk

$$8\cancel{I_0} = \cancel{I_0}(10)^{\frac{P-7.5}{10}}$$
$$8 = 10^{\frac{P-7.5}{10}}$$
$$\log 8 = \log(10)^{\frac{P-7.5}{10}}$$
$$\log 8 = \left(\frac{P-7.5}{10}\right)\underbrace{\log 10}_{=1}$$
$$\log 8 = \frac{P-7.5}{10}$$
$$10\log 8 = P - 7.5$$
$$10\log(8) + 7.5 = P$$
$$\boxed{P = 8.4 \text{ on pH}}$$

Answers:
1. The magnitude 8.3 earthquake is about ***40 times more intense*** than the 6.7 earthquake.
2. Magnitude of the smaller earthquake is 4.5 on Richter scale.
3. The sound of the power saw is about 10^{11} times as intense as that of a leaf rustling.
4. The total loudness is about 83 dB.
5. The solution has a pH of 3.5
6. The pH reading for sea water is about 8.4

Math History

John Napier lived from 1550-1617. He developed logarithms. In those days, logarithms were used mostly to do calculations. By using the laws of logarithms many difficult calculations could be simplified – instead of multiplying, one could use logarithms and then add, or instead of dividing, one could subtract. Here's an example:

Suppose you need to divide 217.39 by 25.461.

Logarithms helped like this:

$$\log_{10}\left(\frac{217.39}{25.461}\right) = \log 217.39 - \log 25.461$$

$$\log_{10}(quotient) = 2.337239563 - 1.405875457$$

$$\log_{10}(quotient) = 0.931364106$$

$$10^{0.931364106} = \text{quotient}$$

$$8.53815640 = \text{quotient}$$

The famous mathematician, Leonhard Euler, studied the number "*e*"

Leonhard Euler lived from 1707-1783. He published 530 books and papers during his lifetime. For the last 16 years of his life he was totally blind, but thanks to his phenomenal memory and ability to concentrate, he continued to generate a lot of mathematics. He would write formulas in chalk on a large slate for his secretary to copy down. He standardized these notations that you may know:

$f(x)$ for function notation

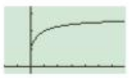
i for the imaginary unit, $\sqrt{-1}$

He came up with this formula, $e^{\pi i} + 1 = 0$, relating five of the most important numbers in mathematics.

The equation below tells us that as x gets huge, the y-values of the graph get closer and closer to the value of e:

$$e = \lim_{x \to \infty}\left(1 + \frac{1}{x}\right)^x$$

Notice that the graph is approaching a horizontal asymptote which is somewhere between 2 and 3. The table of values shows that the y values are getting closer to the actual value of e.



Know these 4 log laws.

1) Product Law: $\log(XY) = \log X + \log Y$

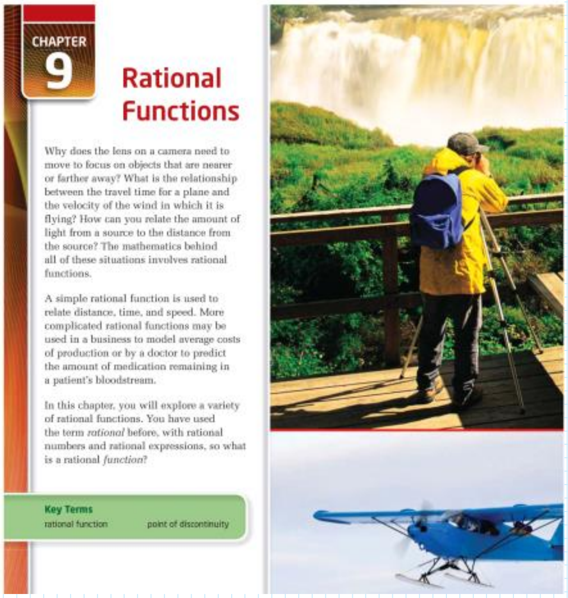
2) Quotient Law: $\log\left(\frac{A}{B}\right) = \log A - \log B$

3) Power Law: $\log(C^D) = D\log C$

4) Change of Base Law: $\log_A(B) = \frac{\log_c B}{\log_c A}$

$$\log_3 17 = \frac{\log_{10} 17}{\log_{10} 3}$$

TB page 428



CHAPTER 9

Rational Functions

Why does the lens on a camera need to move to focus on objects that are nearer or farther away? What is the relationship between the travel time for a plane and the velocity of the wind in which it is flying? How can you relate the amount of light from a source to the distance from the source? The mathematics behind all of these situations involves rational functions.

A simple rational function is used to relate distance, time, and speed. More complicated rational functions may be used in a business to model average costs of production or by a doctor to predict the amount of medication remaining in a patient's bloodstream.

In this chapter, you will explore a variety of rational functions. You have used the term *rational* before, with rational numbers and rational expressions, so what is a rational *function*?

Key Terms

rational function   point of discontinuity

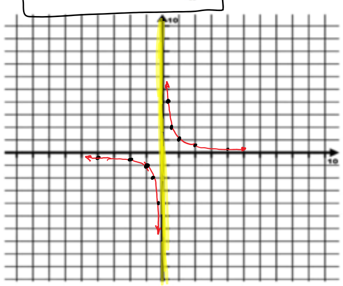
Chapter 9: Rational Functions

9.1 Exploring Rational Functions Using Transformations

Rational function are functions that can be written in the form $y = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.

Rational Functions with Linear Numerators and Denominators

Base rational function, $y = \frac{1}{x}$ ← $x \neq 0$



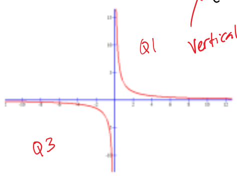
$y = \frac{1}{x}$	
x	y
-4	$\frac{1}{-4}$
-2	$\frac{1}{-2}$
-1	$\frac{1}{-1} = -1$
$-\frac{1}{2}$	$\frac{1}{-\frac{1}{2}} = 1 \cdot \frac{-2}{1} = -2$
$-\frac{1}{4}$	-4
0	$\frac{1}{0}$ = DNE (does not exist)
$\frac{1}{4}$	4
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
4	$\frac{1}{4}$

Characteristics

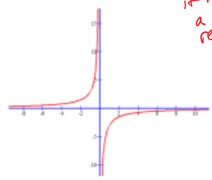
Non-permissible value (NPV)	$x = 0$ is the only NPV
End behavior As $\lvert x \rvert$ becomes very large, what does y do?	y approaches zero
Domain	$\{x \mid x \neq 0, x \in \mathbb{R}\}$
Range	$\{y \mid y \neq 0, y \in \mathbb{R}\}$
Equation of vertical asymptote	$x = 0$
Equation of horizontal asymptote	$y = 0$

In general, these are the ***simplest*** rational function equations, together with their graphs:

$y = \frac{a}{x}$, $a > 0$ $y = a\left(\frac{1}{x}\right)$ ← Vertical stretch



$y = \frac{a}{x}$, $a < 0$ ← if it's negative, it makes a reflection across x-axis



Transformations of $y = \frac{1}{x}$ ← causes translations

For $\boxed{y = \frac{a}{x-h} + k}$, complete the table below.

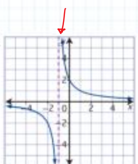
Non-permissible value (NPV)	$x = h$ is the NPV
End behavior As $\lvert x \rvert$ becomes very large, what does y do?	y-values approach "k"
Domain	$\{x \mid x \neq h,\ x \in \mathbb{R}\}$
Range	$\{y \mid y \neq k,\ y \in \mathbb{R}\}$
Equation of vertical asymptote	$x = h$
Equation of horizontal asymptote	$y = k$

TB p 442: 1

$A(x) = \frac{2}{x} - 1$ ← down 1 $B(x) = \frac{2}{x+1}$ ← (all have been vertically stretched by 2)

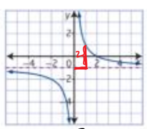
$C(x) = \frac{2}{x-1}$ $D(x) = \frac{2}{x} + 1$

a)



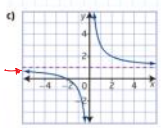
$B(x) = \frac{2}{x+1}$ 1 left

b)



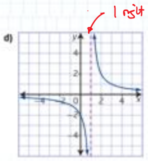
$A(x) = \frac{2}{x} - 1$

c)



$D(x) = \frac{2}{x} + 1$ up 1

d)



1 right

$C(x) = \frac{2}{x-1}$ 1 right

To Try

1. Given the original rational function $y=\frac{1}{x}$ and the transformed function $y=\frac{4}{x+3}-5$:

means the same as $y=4\left(\frac{1}{x+3}\right)-5$

VE by 4   3 left   5 down

a) Complete the tables below. For the first table, give 6 points found on the graph of the ***original function*** $y=\frac{1}{x}$. In the final table, give the image points that result after the transformations have occurred. Write the mapping notation in the heading of that table.

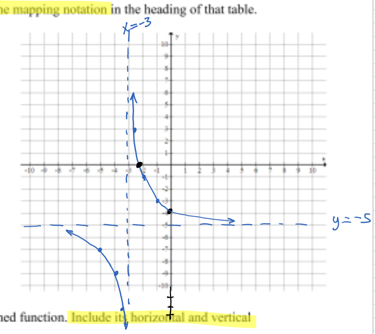
Base function $y=\frac{1}{x}$

x	y
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$

Transformed function $y=\frac{4}{x+3}-5$

x-3	4y-5
-5	-7
-4	-9
$-3\frac{1}{2}$	-13
$-2\frac{1}{2}$	3
-2	-1
-1	-3

mapping: $(x,y)\to(x-3,\,4y-5)$



b) Accurately sketch the final transformed function. Include its horizontal and vertical asymptotes, drawn with dotted lines.

c) Give the equations of the asymptotes.   v.a. $x=-3$ h.a. $y=-5$

d) Use algebra to find the coordinates of the final graph's x-intercept and y-intercept.

x-int, let $y=0$

$y=\frac{4}{x+3}-5$

$0=\frac{4}{x+3}-5$

$(x+3)(5)=\left(\frac{4}{\cancel{x+3}}\right)\cancel{(x+3)}$

$5x+15=4$

$\frac{5x}{5}=\frac{-11}{5}$ $x=\frac{-11}{5}$

$\left(-\frac{11}{5},0\right)$ or $\left(-2\frac{1}{5},0\right)$

y-int, let $x=0$

$y=\frac{4}{0+3}-5$

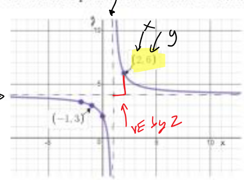
$y=\frac{4}{3}-\frac{5}{1}\left(\frac{3}{3}\right)$

$y=\frac{4}{3}-\frac{15}{3}$

$y=-\frac{11}{3}$

$\left(0,-\frac{11}{3}\right)$ $\left(0,-3\frac{2}{3}\right)$

2. Find the equation of the rational function graphed below.



x=1   y=4   VE by 2

From the asymptotes, we get:

$y=\frac{a}{x-1}+4$ (VE by 2)

$y=\frac{2}{x-1}+4$

We know: $y=\frac{a}{x-1}+4$

substitute in any point we know that's on the graph

$6=\frac{a}{2-1}+4$

$6=\frac{a}{1}+4$

$2=a$

Example

Given the rational function $y=\frac{-4x+3}{x+2}$

Find its NPVs, intercepts, and asymptote equations.

1) NPV: denom $\neq 0$

$x+2\neq 0$

$x\neq -2$

2) x-int, let $y=0$

$(x+2)(0)=\left(\frac{-4x+3}{\cancel{x+2}}\right)\cancel{(x+2)}$

$0=-4x+3$
$+4x \quad +4x$

$\frac{4x}{4}=\frac{3}{4}$

$x=\frac{3}{4}$ $\left(\frac{3}{4},0\right)$

y-int, let $x=0$

$y=\frac{-4x+3}{x+2}$, $y=\frac{-4(0)+3}{0+2}=\frac{3}{2}$

$\left(0,\frac{3}{2}\right)$

3) asymptotes

vertical asymptote – comes from setting denom $=0$ (it's where your NPV is!)

$x+2=0$

$\boxed{x=-2}$

horizontal asymptote

$y=\frac{-4x+3}{x+2}$

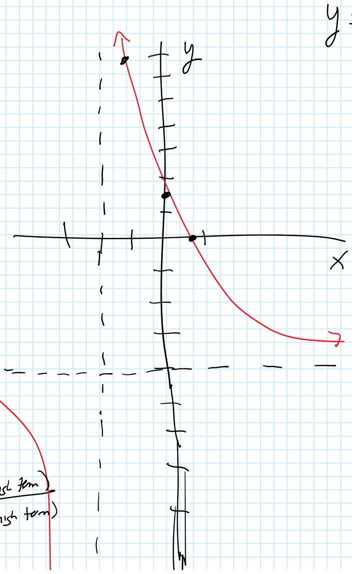
$y=\frac{-4(\text{billion})+3}{\text{billion}+2}$ this makes very little difference

$y=-4$

Look at the equation's terms with the highest degree.

Equation: $y=\frac{\text{coeff (high term)}}{\text{coeff (high term)}}$

$y=\frac{-4x+3}{x+2}$



x	y
-1	$\frac{-4(-1)+3}{1}=\frac{7}{1}=7$
-3	$\frac{-4(-3)+3}{-3+2}=\frac{12+3}{-1}=\frac{15}{-1}=-15$

$(0, \frac{3}{2})$

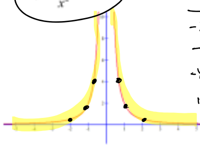
degree.

Equation: $y = \frac{\text{Coeff (high term)}}{\text{Coeff (high term)}}$

-15

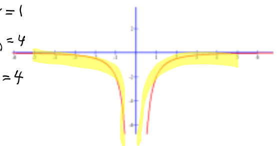
Rational Functions with Quadratic Term in Denominator, no other *x*'s

$y = \frac{a}{x^2},\ a > 0$ $\quad y = \frac{1}{x^2}$



x	y
-2	$\frac{1}{(-2)^2} = \frac{1}{4}$
-1	$\frac{1}{(-1)^2} = \frac{1}{1} = 1$
$-\frac{1}{2}$	$\frac{1}{(-\frac{1}{2})^2} = \frac{1}{(\frac{1}{4})} = 4$
$\frac{1}{2}$	$\frac{1}{(\frac{1}{2})^2} = \frac{1}{(\frac{1}{4})} = 4$
1	
2	

$y = \frac{a}{x^2},\ a < 0$ $\quad y = -\frac{1}{x^2}$



Non-permissible value (NPV)	$x = 0$
Domain	$\{x \mid x \neq 0,\ x \in \mathbb{R}\}$
Range	$y = \frac{a}{x^2},\ a>0$: $\{y \mid y > 0,\ y \in \mathbb{R}\}$ $\quad$ $y = \frac{a}{x^2},\ a<0$: $\{y \mid y < 0,\ y \in \mathbb{R}\}$
Equation of vertical asymptote	$x = 0$
Equation of horizontal asymptote	$y = 0$

Transformations of $y = \frac{1}{x^2}$

For $y = \frac{a}{(x-h)^2} + k$, complete the table below.

Non-permissible value (NPV)	$x \neq h$
Domain	$\{x \mid x \neq h,\ x \in \mathbb{R}\}$
Range	$a>0$: $\{y \mid y > k\}$ $\quad$ $a<0$: $\{y \mid y < k\}$
Equation of vertical asymptote	$x = h$
Equation of horizontal asymptote	$y = k$

Practice

(9.1) TB p 442: 2ac, 3cd, 4ac, 5ac, 6, 7bd, 8, 9, 12, 16

For next class, Thursday, Nov 24

- **Complete Chapter 8 Hand-in**
- **Prepare for the Unit 3 Test (Chapters 7-8, including "e" and natural log)**
- **Can use scientific calculator, graphing calculator, and/or Desmos SCIENTIFIC calculator on this exam.**

Study Suggestions:

- **Complete optional worksheets (posted on website):**
  - **More Solving Practice (Log & Exponential Equations)**
  - **Applications Set-up & Solving**
  - **Chapter 8 Review**
  - **Unit 3 Practice Test**
- **Equation solving:**
  - **TB p 412: 1, 2ac, 3, 4ac, 5, 6, 7acd, 8abd, 13, 16**

  Log Scale questions:
  - **TB p 401: 13bc, 16bc**
  - **TB p 417: 15, 17**
  - **TB p 419: 6, 15**