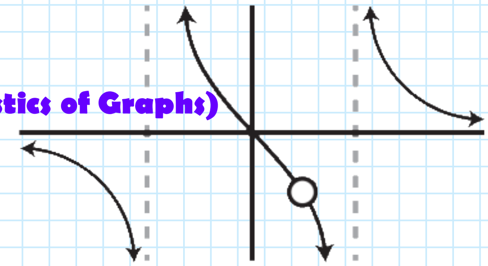
Class_22 June 12 - Characteristics of Rational Function Graphs

Plan For Today:

1. Question about anything from last week? 8.2-part of 9.2
   - **DO TEST 6 ~ 1 HOUR 15 MIN**

2. Continue Chapter 9: Rational Functions
   - 9.1: Rational Function Transformations
   - **9.2: Analysing Rational Functions (Characteristics of Graphs)**
   - 9.3: Graphs and Solving Rational Functions



Calcworkshop.com

5. Work on practice questions from Textbook

Page 452: #4-7, 8ac, 11, 14

Plan Going Forward:

1. Finish working through extra practice & textbook questions from 9.3 and continue working on the Ch. 8 Assignment.
2. You will finish ch9 (9.3) tomorrow and you will start topic 10 on Wednesday.

- CHAPTER 9 ASSIGNMENT DUE ON THURSDAY, JUN 15TH
- TEST 7 ON 9.3-10.4 ON MONDAY, JUNE 12TH
- TOPIC 10 (G) ASSIGNMENT DUE ON TUESDAY, JUNE 20TH

Please let us know if you have any questions or concerns about your progress in this course. The notes from today will be posted at egolfmath.weebly.com after class.
Anurita Dhiman = adhiman@sd35.bc.ca
Susana Egolf = segolf@sd35.bc.ca

Key Ideas for Rational Function Graphs

1) Horizontal Asymptotes

Find the degree of the numerator and denominator.

	Example	
Numerator degree < Denominator degree ***horizontal asymptote equation:*** $y = 0$	$y = \frac{2x}{3x^2+4}$ h.a. $y = 0$	$y = \frac{5}{x} + 3$ shifted up; h.a. $y=3$
Numerator degree = Denominator degree ***horizontal asymptote equation:*** $y = \frac{\text{leading coefficient of num.}}{\text{leading coefficient of denom.}}$	$y = \frac{5x^2-7}{2x^2+3}$ h.a. $y = \frac{5}{2}$	
Numerator degree > Denominator degree Graph will have a ***slant asymptote***	$y = \frac{5x^3}{2x}$ slant asymptote	

2) NPVs, PODs, and vertical asymptotes

Factor numerator and denominator completely.

- Set ***each factor of the denominator = 0***, to get all NPVs.
  - Is there a factor that cancels with a factor in the numerator? It gives the x-value of a POD.
  - Is there a factor that doesn't cancel with a numerator factor? It gives the location of a vertical asymptote.

3) Intercepts

- ***y*-intercepts** – substitute $x = 0$ into the function (either the original or the simplified form) and solve for y
- ***x*-intercepts** - set each factor of the simplified numerator = 0 and solve for x

4) Sketch

- Plot all x-intercepts and y-intercepts
- Show points of discontinuity (PODs) as "holes", using an open circle
- Show all asymptotes as dotted lines.
- Find more points on the graph, as needed, by substituting into its equation.
- Make sure graph does not cross any vertical asymptotes.

Summary of Characteristics from Factored Form

$$y = \frac{C(x-a)(x-c)}{(x-b)(x-c)}$$

$y = C$ Horizontal asymptote

$x-a$ is the location of x-intercept $(a, 0)$

$x-b$ is the vertical asymptote $x = b$

$x-c$ is the POD at $x=c$ (c, y)

= NPVs $x=c$, $x=b$

To try:

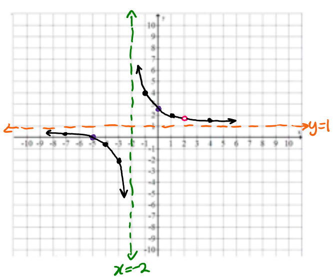
Original Equation	Factored form of equation	List all NPVs and for each one identify if it gives a POD or a vertical asymptote. - Find the (x, y) coordinates of each POD . - Find the equation of each vertical asymptote.	Horizontal asymptote equation or say "Slant"
$y=\frac{2x+10}{x^2+2x-15}$ GCF = 2 AC −15 5, −3 = 2	$y=\frac{2(\cancel{x+5})}{(\cancel{x+5})(x-3)}$ $\boxed{y=\frac{2}{x-3}}$	NPV : $x \neq -5$, $x \neq 3$ POD $x=-5$ $y=\frac{2}{-5-3}$ $=\frac{2}{-8}$ $y=-\frac{1}{4}$ → POD$(-5,-\frac{1}{4})$ asymptote (VA) $x=3$ Domain $\{x \mid x\neq -5, x\neq 3, x\in R\}$ Range $\{y \mid y\neq -\frac{1}{4}, y\neq 0, y\in R\}$	$y=0$ b/c - lower degree in numerator x^1 x^0 - higher degree in denominator x^2 x^1
$y=\frac{2x^2+7x+6}{x^2-2x-8}$ AC 12 +4, +3 = 7 $2x^2+4x+3x+6$ $2x(x+2)+3(x+2)$ −4, +2 = −2	$y=\frac{(2x+3)(\cancel{x+2})}{(x-4)(\cancel{x+2})}$ $\boxed{y=\frac{2x+3}{x-4}}$	NPVs : $x\neq -2$, $x\neq 4$ POD @ $x=-2$ $y=\frac{2(-2)+3}{-2-4}$ $y=\frac{-1}{-6}$ $y=\frac{1}{6}$ $(-2,\frac{1}{6})$ VA $x=4$ $\{x \mid x\neq -2, x\neq 4, x\in R\}$ $\{y \mid y\neq \frac{1}{6}, y\neq 2, y\in R\}$	$y=2$ HA
$y=\frac{x^2+3x-4}{x-1}$ −4 +4, −1 = 3	$y=\frac{(x+4)(\cancel{x-1})}{\cancel{x-1}}$ $\boxed{y=x+4}$	NPV $x\neq 1$ POD at $x=1$ $y=1+4$ $y=5$ $\boxed{(1,5)}$ No VA $\{x \mid x\neq 1, x\in R\}$ $\{y \mid y\neq 5, y\in R\}$.	NO HA (or slant asymptote ok) ← higher degree numerator lower degree denominator

Whiteboards -Characteristics of Rational Functions

Without using technology, accurately sketch the function's graph: $y=\frac{x^2+3x-10}{x^2-4}$

Give the values of the graph's:

- NPVs
- asymptote equations
- coordinates of PODs
- x- and y-intercepts



$y=\frac{(x+5)(\cancel{x-2})}{(x+2)(\cancel{x-2})}$

$y=\frac{x+5}{x+2}$

NPVs $x\neq -2$, $x\neq 2$

VA
$x=-2$

POD
@ $x=2$
$y=\frac{2+5}{2+2}$
$y=\frac{7}{4}$ $(2,\frac{7}{4})$

HA → $y=1$

x-int
$x=-5$
$(-5,0)$

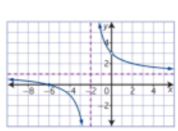
y-int
$y=\frac{0+5}{0+2}$
$y=\frac{5}{2}$
$(0,\frac{5}{2})$

other points
- look for x-coordinates where it would to draw a curve accurately.
- sub into simplified equation to get y-coordinate

x	y
−7	2/5
−4	−1/2
−3	−2
1	2
4	3/2
−1	4

→ $y=\frac{-7+5}{-7+2}$ → $\frac{-2}{-5}$

Example (TB page 453, #7a)
Write the equation of the pictured rational function.



Graph Rational Functions with Holes

If the degree of the numerator < degree of the denominator then horizontal asymptote is at y = 0.

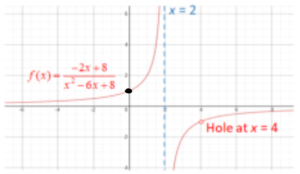
The vertical asymptotes will occur where the denominator equals zero.

If there is a common factor in the numerator and denominator then the graph of a rational function will have a hole when a value of x causes both the numerator and the denominator to equal 0. We can set the common factor to zero and solve for x to find the hole.

Example:

$$f(x)=\frac{-2x+8}{x^2-6x+8}=\frac{-2(x-4)}{(x-4)(x-2)}$$

hole at $x=4$ vertical asymptote at $x=2$



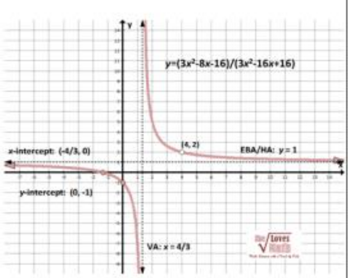
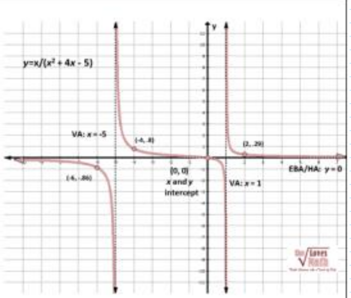
Sketching the Graph of a Rational Function by Hand

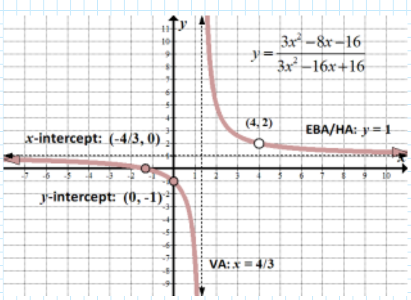
Guidelines for Graphing Rational Functions

1. Write the rational expression in **simplest form**, by factoring the numerator and denominator and dividing out common factors.
2. Find the coordinates of any **"holes"** in the graph.
3. Find and plot the **y-intercept**, if any, by evaluating **f(0).**
4. Find and plot the **x-intercept(s)**, if any, by **finding the zeros of the numerator**.
5. Find the **vertical asymptote(s)**, if any, by **finding the zeros of the denominator**. Sketch these using dashed lines.
6. Find the **horizontal asymptote**, if any, by **comparing the degrees of the numerator and denominator**. Sketch these using dashed lines.
7. Find the **oblique asymptote**, if any, by **dividing the numerator by the denominator using long division.**
8. Plot **5-10 additional points**, including points close to each x-intercept and vertical asymptote.
9. Use **smooth curves** to complete the graph.

Rules for Graphing Rationals	Examples
To get the **end behavior asymptote (EBA)**, you want to compare the degree in the numerator to the degree in the denominator. There can be **at most 1 EBA** and most of the time, these are horizontal. ➢ If the degree (largest exponent) on the **bottom is greater** than the degree on the top, the EBA (which is also a **horizontal asymptote** or **HA**) is $\boldsymbol{y=0}$.	$y=\frac{x+2}{x^2-4}$ Notice that even though we can take out a removable discontinuity ($x+2$), the bottom still has a higher degree than the top, so the HA/EBA is $\boldsymbol{y=0}$.
➢ If the degree on the **top is greater** than the degree on the bottom, there is no EBA/HA. **However, if the degree on the top is one more than the degree on the bottom, than there is a slant (oblique) EBA asymptote, which is discussed below.**	$y=\frac{x^3+2}{x-4}$ No HA/EBA. Vertical asymptote is still $\boldsymbol{x=4}$.
➢ If the degree is the **same on the top and the bottom**, than **divide coefficients of the variables with the highest degree on the top and bottom**; this is the HA/EBA. You can determine this asymptote even without factoring.	$y=\frac{2x^3+2}{3x^3-4}$ Since the degree on the top and bottom are both 3, the HA/EBA is $\boldsymbol{y=\frac{2}{3}}$.
➢ If the degree on the top is **one more** than the degree on the bottom, then the function has a **slant** or **oblique EBA** in the form $y=mx+b$. We have to use **long division** to find this equation. We can just ignore or "throw away" the remainder and just use the linear equation. Weird, huh?	$y=\frac{2x^2+x+1}{x-4}$ $\begin{array}{r} 2x+9 \\ x-4\overline{)2x^2+x+1} \\ \underline{2x^2-8x} \\ 9x+1 \\ \underline{9x-36} \\ \cancel{37} \end{array}$ EBA: $\boldsymbol{y=2x+9}$
➢ (more Advanced) Find the point where any horizontal asymptotes **cross the function** by **setting the function to the horizontal asymptote**, and solving for "**x**". You already have the "**y**" (from the HA equation).	**Q:** Where does $y=\frac{-x^2+x}{x^2+x-12}$ **intersect** its EBA? **A:** Note that the EBA is $y=\frac{-1}{1}=-1$. Now set $\frac{-x^2+x}{x^2+x-12}=-1$ and cross multiply: $-x^2+x=-1(x^2+x-12)$; $\boldsymbol{x=6}$. So the point where the function intersects the EBA is **(6, −1).**

Rules for Graphing Rationals	Examples
First factor both the numerator and denominator, and **cross out any factors** in both the numerator and denominator. ➢ If any of these factors contain variables, these are **removable discontinuities**, or "**holes**" and will be little circles on the graphs. The idea is that if you cross out a polynomial, you can't forget that it was in the denominator and can't "legally" be set to 0. (We will see graph later.) The **domain** of a rational function is all real numbers, except those that make the denominator equal zero, as we saw earlier. (Note that if after you cross out factors, you still have that same factor on the bottom, the "hole" will turn into a **vertical asymptote**; follow the rules below).	$y=\frac{x^2-5x+6}{x-3}=\frac{\cancel{(x-3)}(x-2)}{\cancel{(x-3)}}=x-2$ This function reduces to the line $y=x-2$ with a **removable discontinuity** (a little circle on the graph) where $x=2$ and $y=(2)-2=0$ (plug 2 in for ***y*** in original or reduced fraction). So the hole is at (2, 0). Domain is $(-\infty,3)\cup(3,\infty)$, since a 3 would make the denominator = 0. It's like we have to "skip over" the 3 with interval notation.
To get **vertical asymptotes or VAs:** ➢ After determining if there are any holes in the graph, factor (if necessary) what's left in the **denominator** and set the factors to 0. For any value of ***x*** where these factors could be 0, this creates a vertical asymptote at "***x =*** " for these values. Note: There could a multiple number of vertical asymptotes, or no vertical asymptotes. Don't forget to include the factors with "***x***" alone (***x* = 0** is the vertical asymptote).	$y=\frac{x^2-5x+6}{x(x^2-9)}=\frac{\cancel{(x-3)}(x-2)}{x\cancel{(x-3)}(x+3)}=\frac{x-2}{x(x+3)}$ Vertical asymptotes occur when $(x-0)=0$ or $(x+3)=0$, or $x=0$ or $x=-3$. Domain is $(-\infty,-3)\cup(-3,0)\cup(0,3)\cup(3,\infty)$, since anything that could make the denominator 0 (even a hole) can't be included. So we have to "skip over" –3, 0, and 3.

Steps		Graph
$y=\frac{3x^2-8x-16}{3x^2-16x+16}$ Factor: $y=\frac{(3x+4)\cancel{(x-4)}}{(3x-4)\cancel{(x-4)}}$ **Removable Discontinuity or Hole:** ***x*** = 4, plug in 4 for ***x*** to get ***y*** = 2. RD is (4, 2). **VA:** Set denominator to 0 after removing the hole: $3x-4=0$ $x=\frac{4}{3}$.	**EBA/HA:** Since the degree is the same on the top and bottom (both are 2), we take the coefficients and divide them: $y=\frac{3}{3}=1$. ***x*-intercept (root):** Set ***y*** (or top) to 0: $3x+4=0$; $x=-\frac{4}{3}$ $\left(-\frac{4}{3},0\right)$ ***y*-Intercept:** Set ***x*** to 0: $y=\frac{3(0)+4}{3(0)-4}=-1$ $(0,-1)$ **Domain:** Can't be any value of ***x*** that makes the bottom zero: $(-\infty,\frac{4}{3})\cup(\frac{4}{3},4)\cup(4,\infty)$	
$y=\frac{x}{x^2+4x-5}$ Factor: $y=\frac{x}{(x+5)(x-1)}$ **VA:** Set denominator to 0 after factoring; we have 2 of them: $x=-5$, $x=1$. **EBA/HA:** Since the degree on the top (1) is less than the degree on the bottom (2), the EBA or VA is ***y*** = 0.	***x*-intercept (root):** Set ***y*** (or top) to 0: $x=0$. $(0,0)$ ***y*-Intercept:** Set ***x*** to 0: $y=\frac{(0)}{(0)^2+4(0)-5}=0$: $(0,0)$ **"T-chart":** Try some points around the vertical asymptotes: $x=-6$, $y=-.86$ $x=-4$, $y=.8$ $x=2$, $y=.29$ **Domain:** Can't be any value of ***x*** that makes the bottom zero: $(-\infty,-5)\cup(-5,1)\cup(1,\infty)$.	



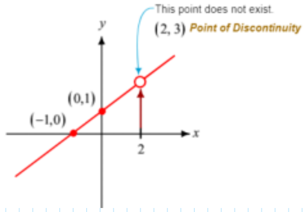
Rules for Asymptotes

Exponent of highest degree term in denominator larger than highest degree term in numerator then Horizontal Asymptote is
$y = 0$

Exponent values of highest degree of terms in numerator and denominator the same then Horizontal Asymptote is
$y =$ ratio of their coefficients

2.2 Point of Discontinuity

$$f(x) = \frac{x^2 - x - 2}{x - 2}$$



C_23 Characteristics of Rational Functions

Tuesday, November 26, 2019 6:25 PM

C_23 Characteristics of Rational Functions

(Solutions at right)

Characteristics of Rational Functions

For each rational function find:
a. the simplified equation
b. coordinates of any PODs
c. vertical asymptote equation(s)
d. horizontal asymptote equation
e. y-intercept
f. x-intercepts

1. $f(x)=\frac{x^2+x-2}{x^2-x-6}$

2. $f(x)=\frac{2x^2}{x^2-1}$

3. $f(x)=\frac{3}{x-2}$

4. $f(x)=\frac{2x-1}{x}$

Characteristics of Rational Functions

For each rational function find:
a. the simplified equation
b. coordinates of any PODs
c. vertical asymptote equation(s)
d. horizontal asymptote equation
e. y-intercept
f. x-intercepts

1. $f(x)=\frac{x^2+x-2}{x^2-x-6} = \frac{(x-1)\cancel{(x+2)}}{\cancel{(x+2)}(x-3)}$

a) $f(x) = \frac{x-1}{x-3}$, $x \neq -2$

b) POD: $x=-2$, $y=\frac{-2-1}{-2-3}=\frac{-3}{-5}$ $\left(-2, \frac{3}{5}\right)$

c) vert. asympt: $x=3$

d) hor. asymp: $y=1$ (degree top = degree bottom $y=\frac{1}{1}=1$)

e) y-int: $y=\frac{0-1}{0-3}$, $y=\frac{-1}{-3}=\frac{1}{3}$ $\left(0, \frac{1}{3}\right)$

f) x-int: $0=\frac{x-1}{x-3} \Rightarrow x-1=0$, $x=1$ $(1,0)$

2. $f(x)=\frac{2x^2}{x^2-1} = \frac{2x^2}{(x+1)(x-1)}$

a) $f(x) = \frac{2x^2}{(x+1)(x-1)}$

b) none

c) v.a. $x=-1$, $x=1$

d) h.a. $y=2$ (degree top = degree bottom $y=\frac{2}{1}=2$)

e) y-int: $y=\frac{2(0)^2}{(0+1)(0-1)}=\frac{0}{-1}$ $(0,0)$

f) x-int: $0=\frac{2x^2}{(x+1)(x-1)}$

$\frac{0}{2} = \frac{2x^2}{2}$ $\rightarrow x^2=0$

$(0,0)$

3. $f(x)=\frac{3}{x-2}$

a) $f(x) = \frac{3}{x-2}$

b) POD none

c) v.a. $x=2$

d) h.a. $y=0$ (degree top smaller than degree bottom $\Rightarrow y=0$)

e) y-int: $y=\frac{3}{0-2}=\frac{3}{-2}$ $\left(0, -\frac{3}{2}\right)$

f) x-int: $0=\frac{3}{x-2}$

$0=3$

not possible!

$\underline{\underline{\text{no}}}$ x-int.

4. $f(x)=\frac{2x-1}{x}$

a) $f(x)=\frac{2x-1}{x}$

b) POD none

c) v.a. $x=0$

d) h.a. $y=2$ (degree top = degree bottom $y=\frac{2}{1}=2$)

e) y-int: $y=\frac{2(0)-1}{0}$

$y=\frac{-1}{0}$ $\underline{\underline{\text{no}}}$ y-intercept

f) x-int: $0=\frac{2x-1}{x}$

$0=2x-1$

$1=2x$

$x=\frac{1}{2}$ $\left(\frac{1}{2}, 0\right)$

For each rational function find:
a. the simplified equation
b. coordinates of any PODs
c. vertical asymptote equation(s)
d. horizontal asymptote equation
e. y-intercept
f. x-intercepts

5. $f(x)=\frac{x^2+x-12}{x^2-9}$

6. $f(x)=\frac{x^2-x-2}{x-1}$

7. $f(x)=\frac{x+1}{x^2+3x+2}$

8. $f(x)=\frac{x^2-9}{x^2-2x-3}$

For each rational function find:
a. the simplified equation
b. coordinates of any PODs
c. vertical asymptote equation(s)
d. horizontal asymptote equation
e. y-intercept
f. x-intercepts

5. $f(x)=\frac{x^2+x-12}{x^2-9} = \frac{\cancel{(x-3)}(x+4)}{\cancel{(x-3)}(x+3)}$

a) $f(x)=\frac{x+4}{x+3}$, $x \neq 3$

b) POD $x=3$

$y=\frac{3+4}{3+3}=\frac{7}{6}$ $\left(3, \frac{7}{6}\right)$

c) v.a. $x=-3$

d) h.a. $y=1$ (degree top = degree bottom $y=\frac{1}{1}=1$)

e) y-int: $y=\frac{0+4}{0+3}=\frac{4}{3}$ $\left(0, \frac{4}{3}\right)$

f) x-int: $0=\frac{x+4}{x+3}$, $0=x+4$, $x=-4$

$(-4,0)$

6. $f(x)=\frac{x^2-x-2}{x-1} = \frac{(x+1)(x-2)}{x-1}$

a) $f(x)=\frac{(x+1)(x-2)}{x-1}$

b) no PODs

c) v.a. $x=1$

d) h.a. slant (degree top BIGGER than degree bottom $\Rightarrow$ slant asymptote)

e) y-int: $y=\frac{(0+1)(0-2)}{0-1}=\frac{1(-2)}{-1}=\frac{-2}{-1}$

$(0,2)$

f) X-int: $0=\frac{(x+1)(x-2)}{x-1}$

$0=(x+1)(x-2) \Rightarrow$ $(-1,0)$ and $(2,0)$

7. $f(x)=\frac{x+1}{x^2+3x+2} = \frac{\cancel{x+1}}{\cancel{(x+1)}(x+2)}$

a) $f(x)=\frac{1}{x+2}$, $x \neq -1$

b) POD $x=-1$

$y=\frac{1}{-1+2}=\frac{1}{1}=1$ $(-1, 1)$

c) v.a. $x=-2$

d) h.a. $y=0$ (degree top smaller than degree bottom $\Rightarrow y=0$)

e) y-int: $y=\frac{1}{0+2}$

$y=\frac{1}{2}$ $\left(0, \frac{1}{2}\right)$

f) x-int: $0=\frac{1}{x+2}$

$0=1$

not true $\Rightarrow$ $\underline{\underline{\text{no}}}$ x-intercept

8. $f(x)=\frac{x^2-9}{x^2-2x-3} = \frac{(x+3)\cancel{(x-3)}}{(x+1)\cancel{(x-3)}}$

a) $f(x)=\frac{x+3}{x+1}$, $x \neq 3$

b) POD $x=3$

$y=\frac{3+3}{3+1}=\frac{6}{4}$, $y=\frac{3}{2}$

$\left(3, \frac{3}{2}\right)$

c) v.a. $x=-1$

d) h.a. $y=1$ (degree top = degree bottom $y=\frac{1}{1}=1$)

e) y-int: $y=\frac{0+3}{0+1}=\frac{3}{1}$ $(0,3)$

f) x-int: $0=\frac{x+3}{x+1}$

$0=x+3$

$(-3,0)$

$0+1$

f) x-int: $0 = \frac{1}{x+2}$

$0 = 1$

not true $\Rightarrow$ no x-intercept

f) x-int: $0 = \frac{x+3}{x+1}$

$0 = x+3$

$x = -3$

$(-3, 0)$