Class_24 Dec 1 - Geometric Sequences and Series

Sunday, November 20, 2022   3:21 PM

Tonight's Class:

- **Geometric Sequences (G.1)**
- **Geometric Series (G.2)**

Coming up next Wednesday-Thursday, "Ugly Xmas Sweater Days" at LEC

G.1 Geometric Sequences and Series

A sequence of numbers is a list of numbers in a specific order. They are referred to as the ***terms*** in the sequence. What is the next term for each sequence below?

1, 2, 4, 8, 16, 32, 64, 128

5, 10, 15, 20, 25, 30, 35

1, 4, 9, 16, 25, 36, 49, 64

1, 1, 2, 3, 5, 8, 13, 21, 34 Fibonacci

There are many types of sequences. The ones shown below are all the same kind. What ~~do they have in common?~~

Multiplying by a specific number generate the next term in the sequence.

6, 12, 24, 48. $\times 2$

$16, 4, 1, \frac{1}{4}, \frac{1}{16}...$ $\times \frac{1}{4}$

$-5, 15, -45, 135...$ $\times -3$

$-3, -12, -48, -192...$ $\times 4$

All are numbers ...

Geometric sequences are created by multiplying each term after the first one by a constant number which is called the ***common ratio***. In formulas, they use the letter r to stand for the common ratio.

> How can you figure out the value of r?
>
> Divide any term by the one immediately before it, in the sequence.

We use the notation t_n to identify terms in a sequence.
In the sequence beginning 3, 6, 12, 24, 48 . . .

$t_3 = 12$, because 12 is the third term of the sequence

$t_5 = 48$, because 48 is the fifth term of the sequence

$t_4 = 24$

The **first term** in the sequence could be referred to correctly as t_1, but often we refer to it as "a". In the sequence 3, 6, 12, 24, 48 . . . we say $a = 3$.

To try:
Find these terms, for the sequence that begins: 3, 6, 12, 24, 48 . . .

a) $t_6 =$ $96 = 3(2)^5 = 96$

3×2 a bunch of times

b) $t_8 =$ $384 = 3(2^7) = 384$

c) $t_{26} =$ $3\underbrace{(2\times 2\times ---- \times 2)}_{25 \text{ times}} = 3(2)^{25} = \boxed{100,663,296}$

For any geometric sequence the n^{th} term, t_n, is given by the formula:

$t_n = ar^{n-1}$ where a = first term
r = common ratio, $r \neq 0$
n = the term number

To try:

1. Which sequences are geometric? If a sequence is geometric, state its common ratio, r, and give the next 3 terms of the sequence.

a) 5, 10, 15, 20 . . .

not geometric

b) 2, −4, 8, −16, 32...

$r = \frac{-4}{2} = -2$

$32 \times -2 = -64$, 128, -256

c) 8, 2, ½ , . . .

$r = \frac{2}{8} = \frac{1}{4}$

$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$, $\frac{1}{32}$, $\frac{1}{128}$

d) x, $5x^2$, $25x^3$...

$r = \frac{5x^2}{x}$

$r = 5x$

$125x^4$, $625x^5$, $3125x^6$

2. For each geometric sequence, find the requested value.

a) 8, 2, ½ , . . . Write the defining statement for this sequence in the form $t_n = ar^{n-1}$

$t_n = ar^{n-1}$ $\quad t_n = 8\left(\frac{1}{4}\right)^{n-1}$ $\quad$ not ~~$t_n = 2^{n-1}$~~

b) 20, 10, 5 . . . find t_6

$r = \frac{10}{20} = \frac{1}{2}$

$t_6 = 20\left(\frac{1}{2}\right)^5 = \frac{20}{1}\left(\frac{1}{32}\right) = \frac{20}{32}$ or $\boxed{\frac{5}{8} \text{ OR } 0.625}$

c) 21, −42, 84 . . . find t_{10}

$r = \frac{-42}{21} = -2$

$t_{10} = 21(-2)^9 = \boxed{-10\,752}$

d) $2x^2$, $4x^3$, $8x^4$... find t_9.

$r = \frac{4x^3}{2x^2} = 2x$

$t_9 = ar^8$
$= (2x^2)(2x)^8$
$= (2x^2)(256x^8)$
$= 512x^{10}$

3a) Consider the geometric sequence 2, −6, 18, −54, ... , $\underbrace{13\ 122}_{\text{term, } t_n}$

13 122 is a term in this sequence. What term number is it?

$r = \frac{-6}{2} = -3$

$t_n = ar^{n-1}$

$\frac{13122}{2} = \frac{\cancel{2}}{\cancel{2}}(-3)^{n-1}$

$6561 = (-3)^{n-1}$

$(-3)^8 = (-3)^{n-1}$

$\Rightarrow 8 = n-1$

$\boxed{n = 9}$

b) Consider the geometric sequence 1.25, 5, 20...

327 680 is a term in this sequence. What term number is it? (Don't brute-force it!)

$r = \frac{20}{5}$
$r = 4$

$t_n = ar^{n-1}$

$\frac{327\,680}{1.25} = \frac{(\cancel{1.25})}{\cancel{1.25}}(4)^{n-1}$

$262\,144 = 4^{n-1}$

$\log 262144 = \log 4^{n-1}$

$\frac{\log 262144}{\log 4} = \frac{(n-1)\cancel{\log 4}}{\cancel{\log 4}}$

$n-1 = \frac{\log(262\,144)}{\log 4}$

$n-1 = 9$

$\boxed{n = 10}$

OR

$4^9 = 4^{n-1}$

$\Rightarrow 9 = n-1$

$n = 10$

4 12 36 108 $\underset{t_5}{324}$ __ __ $\underset{t_8}{8748}$

4a) In a geometric sequence, $t_5 = 324$ and $t_8 = 8748$. Find the first two terms.

$\frac{324(r^3)}{324} = \frac{8748}{324}$

$r^3 = 27$

$r = 3$

Use r, and divide to go backwards:

$\boxed{t_1 \text{ or } a = 4 \quad t_2 = 12}$

Another way to do it:

$t_8 = 8748$

$t_5 = 324$

$\boxed{t_n = ar^{n-1}}$

$t_8 = ar^7 = 8748$

$t_5 = ar^4 = 324$

$\frac{ar^7}{ar^4} = \frac{8748}{324}$

$r^3 = 27$

$r = 3$

$ar^4 = 324$

$a(3)^4 = 324$

$\frac{a(81)}{81} = \frac{324}{81}$

$\boxed{a = 4}$

first term

2nd term: $a \times r = 4 \times 3$

$\boxed{2^{nd} \text{ term} = 12}$

b) In a geometric sequence, $t_2 = 18$ and $t_5 = 2.25$. Determine the value of t_9.

36 $\underset{t_2}{18}$ __ __ $\underset{t_5}{2.25}$ __ __ __ t_9

$\frac{18r^3}{18} = \frac{2.25}{18}$

$\sqrt[3]{r^3} = \sqrt[3]{0.125}$

$r = 0.5$ or $\frac{1}{2}$

$t_9 = ar^8$

$= 36\left(\frac{1}{2}\right)^8$

$= 0.140625 = \frac{9}{64}$

6. Between the Canadian censuses in 2001 and 2006, the number of people who could speak in Cree had increased by 7%. In 2006, 87 285 people could converse in Cree. Assume the 5-year increase continues to be 7%. To the nearest hundred, how many people will be able to converse in Cree in 2036?

$1 + 0.07$

$r = 1.07$

#people	87,285	87285(1.07)					?
year	2006	2011	2016	2021	2026	2031	2036
	t_1						

$t_n = ar^{n-1}$

$t_7 = 87285(1.07)^6$

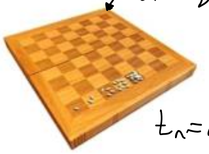
$= 130\,991.2488$

$\doteq 131\,000$ people

sum of terms in a sequence

G. 2 Geometric Series

A chessboard has 64 squares on it. Legend has it that the inventor of the game asked for the following reward from the King: "One grain of wheat on the first square, two on the second, 4 on the third, and so on, doubling the amount on every square until the board is complete."



64th square

$t_n = ar^{n-1}$

How many grains of wheat did the inventor ask for?

$$\underbrace{1+2+4+8+\dots+2^{63}}_{64 \text{ terms}}$$

$t_{64} = 1(2)^{63}$

A ***geometric series*** is the **sum** of the first n terms of a geometric sequence, written as S_n. = sum of n terms

S_3 - is the sum of the first 3 terms of a geometric sequence: $S_3 = t_1 + t_2 + t_3$

For the sequence above, $S_3 = 1 + 2 + 4 = 7$

S_4 - is the sum of the first 4 terms of a geometric sequence: $S_4 = t_1 + t_2 + t_3 + t_4$

For the chessboard sequence, $S_4 = 1 + 2 + 4 + 8 = 15$

For this question, we need to know the value of S_{64}

Here's a way to find the value of a sum of n terms for any geometric series, S_n

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cancel{\dots} + \cancel{ar^{n-2}} + \cancel{ar^{n-1}}$$
$$-rS_n = \quad \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cancel{\dots} + \cancel{ar^{n-2}} + \cancel{ar^{n-1}} + ar^n$$

$S_n - rS_n = a - ar^n$

Subtracting, we get: $S_n - rS_n = a - ar^n$

Factor: $S_n(1-r) = a(1-r^n)$

Divide: $S_n = \dfrac{a(1-r^n)}{1-r}$

$S_n = \dfrac{a(1-r^n)}{1-r}$

$1+2+4+\dots$

$r=2$

$S_{15} = \dfrac{1(1-2^{15})}{1-2} = \dfrac{(1-2^{15})}{-1} = \dfrac{-32\,767}{-1} = \boxed{32\,767}$

$S_{64} = \dfrac{1(1-2^{64})}{1-2} \doteq 1.844 \times 10^{19}$ (quintillion!)

$= 18{,}446{,}744{,}073{,}709{,}551{,}615$ grains of wheat!!

For any geometric sequence, the sum of the first n terms is found using:

$$S_n = \frac{a(1-r^n)}{1-r}$$

a = first term
r = common ratio, $r \neq 1$
n = number of terms to add up

Another form of the formula: $S_n = \dfrac{a - lr}{1-r}$ $\quad l$ = the last term

To Try:

$a=4 \quad r = \frac{12}{4} = 3 \quad n = 9$

1. Given the geometric series: $4 + 12 + 36 + \dots$ find the sum of the first 9 terms.

$S_9 = \dfrac{4(1-3^9)}{1-3} = \boxed{39\,364}$

1. Given the geometric series: 4 + 12 + 36 + . . . Find the sum of the first 9 terms.

$$S_9 = \frac{4(1-3^9)}{1-3} = \boxed{39\,364}$$

2. The sum of the first 12 terms of a geometric series is $\underbrace{24\,570}_{S_{12}}$. The common ratio is -2. $=r$
What is the first term? $=a$

$n = 12$

$$\boxed{S_n} = \frac{a(1-r^n)}{1-r}$$

$$24\,570 = \frac{a(1-(-2)^{12})}{1-(-2)}$$

$$24\,570 = \frac{a(1-4096)}{3}$$

$$3 \times 24\,570 = \frac{a(-4095)}{\cancel{3}} \times \cancel{3}$$

$$\frac{73\,710}{-4095} = \frac{-4095a}{-4095}$$

$$\boxed{a = -18}$$

3a) Find the sum of this series: 0.025 + 0.075 + . . . + 54.675

$l =$ last term

$$S_n = \frac{a - lr}{1-r}$$

$$r = \frac{0.075}{0.025} = 3$$

$$S_n = \frac{0.025 - (54.675)(3)}{1-3}$$

$$\boxed{S_n = 82}$$

b) Use an algebraic method to determine how many terms there are in the above sum.

$$t_n = ar^{n-1}$$

$$\frac{54.675}{0.025} = \frac{\cancel{0.025}(3)^{n-1}}{\cancel{0.025}}$$

$$2187 = 3^{n-1}$$

$$3^7 = 3^{n-1}$$

$$\Rightarrow 7 = n-1$$

$$\boxed{n = 8}$$

4. An oil well produced 10 000 barrels of oil in the first month of production. Each month production is reduced by 10%. What is the total amount of barrels of oil produced by this oil well in its first two years of production?

$10\,000,\ 10\,000(0.9),\ \ldots$

$r = 0.9$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_{24} = \frac{10\,000(1-0.9^{24})}{1-0.9}$$

$= 92\,023$ barrels

Worksheets

- **More Sequences & Series Practice (optional)**

Tuesday, Dec 6

- **Chapter 9 Test**
- **Chapter 9 hand-in due**
- **We'll finish up the unit**

Thursday, Dec 8

- **Geometric Sequences & Series hand-in due**
- **Unit 4 Test**
- **LAST day to hand in any assignments**

Tuesday, Dec 13

Optional class, for Unit 4 re-write