Class_24 June 14 - Geometric Sequences & Series

Plan For Today:

1. Question about anything from last class (9.2-9.3)?
2. Start Topic 10: Geometric Sequences & Series
    - **G.1 Geometric Sequences**
    - **G.2 Geometric Series**
    - G.3 Infinite Geometric Series
    - G.4 Sigma Notation
3. Work on practice questions in handouts.

Plan Going Forward:

1. Practice working through G.1 to G.2 questions.
2. You will review these topics and do G.3 tomorrow. Plan is to finish G.4 on Monday and review for Test 7.

- **CHAPTER 9 ASSIGNMENT DUE ON THURSDAY, JUN 15TH**
- **TOPIC 10 (G) ASSIGNMENT DUE ON TUESDAY, JUNE 20TH**
- **TEST 7 ON 9.3-10.4 ON TUESDAY, JUNE 20TH**

Please let us know if you have any questions or concerns about your progress in this course. The notes from today will be posted at egolfmath.weebly.com after class.
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G.1 Geometric Sequences and Series

A sequence of numbers is a list of numbers in a specific order. They are referred to as the *terms* in the sequence. What is the next term for each sequence below?

1, 2, 4, 8, 16, 32, 64, 128 | 1, 4, 9, 16, 25, 36, 49, 64
5, 10, 15, 20, 25, 30, 35 | 1, 1, 2, 3, 5, 8, 13, 21, 34 Fibonacci

There are many types of sequences. The ones shown below are all the same kind. What do they have in common?

Multiplying by a specific number to generate the next term in the sequence.

6, 12, 24, 48... $\times 2$ | $-5, 15, -45, 135...$ $\times -3$
16, 4, 1, $\frac{1}{4}$, $\frac{1}{16}$... $\times \frac{1}{4}$ | $-3, -12, -48, -192...$ $\times 4$

All are numbers ...

Geometric sequences are created by multiplying each term after the first one by a constant number which is called the ***common ratio***. In formulas, they use the letter ***r*** to stand for the common ratio.

> How can you figure out the value of r?
>
> Divide any term by the one immediately before it, in the sequence.

We use the notation t_n to identify terms in a sequence.
In the sequence beginning 3, 6, 12, 24, 48 . . .

$t_3 = 12$, because 12 is the third term of the sequence
$t_5 = 48$, because 48 is the fifth term of the sequence

$t_4 = 24$

The **first term** in the sequence could be referred to correctly as t_1, but often we refer to it as "a". In the sequence 3, 6, 12, 24, 48 . . . we say $a = 3$.

To try:
Find these terms, for the sequence that begins: 3, 6, 12, 24, 48 , . .

a) $t_6 =$ $96 = 3(2)^5 = 96$

3×2 a bunch of times

b) $t_8 =$ $384 = 3(2^7) = 384$

c) $t_{26} =$ $3(\underbrace{2 \times 2 \times \ldots \times 2}_{25 \text{ times}}) = 3(2)^{25} = \boxed{100,663,296}$

> For any geometric sequence the n^{th} term, t_n, is given by the formula:
>
> $t_n = ar^{n-1}$ where a = first term
> r = common ratio, $r \neq 0$
> n = the term number

To try:

1. Which sequences are geometric? If a sequence is geometric, state its common ratio, r, and give the next 3 terms of the sequence.

a) 5, 10, 15, 20 . . .
not geometric

b) 2, –4, 8, –16, 32...
$r = \frac{-4}{2} = -2$
$32 \times -2 = -64$, 128, -256

c) 8, 2, ½ , . . .
$r = \frac{2}{8} = \frac{1}{4}$
$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$, $\frac{1}{32}$, $\frac{1}{128}$

d) x, $5x^2$, $25x^3$...
$r = \frac{5x^2}{x}$
$r = 5x$
$125x^4$, $625x^5$, $3125x^6$

2. For each geometric sequence, find the requested value.

a) 8, 2, ½ , . . . Write the defining statement for this sequence in the form $t_n = ar^{n-1}$

$r = \frac{2}{8} = \frac{1}{4}$ $\frac{1}{32}$ / $\frac{1}{128}$ $r = \frac{5x^2}{x}$ $r = 5x$

2. For each geometric sequence, find the requested value.

a) 8, 2, ½ , . . . Write the defining statement for this sequence in the form $t_n = ar^{n-1}$

$t_n = ar^{n-1}$ $t_n = 8\left(\frac{1}{4}\right)^{n-1}$ not ~~$t_n = 2^{n-1}$~~

b) 20, 10, 5 . . . find t_6

$r = \frac{10}{20} = \frac{1}{2}$ $t_6 = 20\left(\frac{1}{2}\right)^5 = \frac{20}{1}\left(\frac{1}{32}\right) = \frac{20}{32}$ or $\boxed{\frac{5}{8} \text{ OR } 0.625}$

c) 21, −42, 84 . . . find t_{10}

$r = \frac{-42}{21} = -2$ $t_{10} = 21(-2)^9 = \boxed{-10\,752}$

d) $2x^2$, $4x^3$, $8x^4$... find t_9.

$r = \frac{4x^3}{2x^2} = 2x$ $t_9 = ar^8 = (2x^2)(2x)^8 = (2x^2)(256x^8) = 512x^{10}$

3a) Consider the geometric sequence 2, −6, 18, −54, ... , $\underbrace{13\ 122}_{\text{term}}$

13 122 is a term in this sequence. What term number is it? t_n $r = \frac{-6}{2} = -3$

$t_n = ar^{n-1}$

$\frac{13122}{2} = \frac{\cancel{2}}{\cancel{2}}(-3)^{n-1}$

$6561 = (-3)^{n-1}$

$(-3)^8 = (-3)^{n-1}$

$\Rightarrow 8 = n-1$

$\boxed{n = 9}$

b) Consider the geometric sequence 1.25, 5, 20...

327 680 is a term in this sequence. What term number is it? (Don't brute-force it!)

$r = \frac{20}{5}$ $r = 4$

$t_n = ar^{n-1}$

$\frac{327\,680}{1.25} = \frac{(\cancel{1.25})}{\cancel{1.25}}(4)^{n-1}$

$262\,144 = 4^{n-1}$

$\log 262144 = \log 4^{n-1}$

$\frac{\log 262144}{\log 4} = \frac{(n-1)\cancel{\log 4}}{\cancel{\log 4}}$

$n-1 = \frac{\log(262\,144)}{\log 4}$

$n-1 = 9$

$\boxed{n = 10}$

OR

$4^9 = 4^{n-1}$

$\Rightarrow 9 = n-1$

$n = 10$

4 12 36 108 324 ___ ___ 8748
t_5 t_8

4a) In a geometric sequence, $t_5 = 324$ and $t_8 = 8748$. Find the first two terms.

$\frac{\cancel{324}(r^3)}{\cancel{324}} = \frac{8748}{324}$

$r^3 = 27$

$r = 3$

Use r, and divide to go backwards:

$\boxed{\begin{array}{l} t_1 \text{ or } a = 4 \\ t_2 = 12 \end{array}}$

Another way to do it:

$t_8 = 8748$

$t_5 = 324$

$\boxed{t_n = ar^{n-1}}$

$t_8 = ar^7 = 8748$

$t_5 = ar^4 = 324$

$\frac{\cancel{a}r^7}{\cancel{a}r^4} = \frac{8748}{324}$

$r^3 = 27$

$r = 3$

$ar^4 = 324$

$a(3)^4 = 324$

$\frac{a(81)}{81} = \frac{324}{81}$

$\boxed{a = 4}$ first term

2nd term: $a \times r = 4 \times 3$

$\boxed{2^{nd} \text{ term} = 12}$

b) In a geometric sequence, $t_2 = 18$ and $t_5 = 2.25$. Determine the value of t_9.

36 18 ___ ___ 2.25 ___ ___ ___ t_9
t_2 t_5

$\frac{18r^3}{18} = \frac{2.25}{18}$

$\sqrt[3]{r^3} = \sqrt[3]{0.125}$

$r = 0.5$ or $\frac{1}{2}$

$t_9 = ar^8$

$= 36\left(\frac{1}{2}\right)^8$

$= 0.140625 = \frac{9}{64}$

6. Between the Canadian censuses in 2001 and 2006, the number of people who could speak in Cree had increased by 7%. In 2006, 87 285 people could converse in Cree. Assume the 5-year increase continues to be 7%. To the nearest hundred, how many people will be able to converse in Cree in 2036?

$1 + 0.07$

$r = 1.07$

# people	87,285	87285(1.07)					?
year	2006 (t_1)	2011	2016	2021	2026	2031	2036

$t_n = ar^{n-1}$

$t_7 = 87285(1.07)^6$

$= 130\,991.2488$

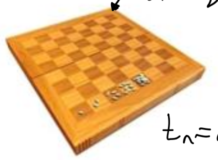
$\doteq 131\,000$ people

sum of terms in a sequence

G. 2 Geometric Series

A chessboard has 64 squares on it. Legend has it that the inventor of the game asked for the following reward from the King: "One grain of wheat on the first square, two on the second, 4 on the third, and so on, doubling the amount on every square until the board is complete."

64th square



$t_n = ar^{n-1}$

How many grains of wheat did the inventor ask for?

$$\underbrace{1 + 2 + 4 + 8 + \ldots + 2^{63}}_{64 \text{ terms}}$$

$t_{64} = 1(2)^{63}$

A ***geometric series*** is the **sum** of the first n terms of a geometric sequence, written as S_n. = sum of n terms

S_3 - is the sum of the first 3 terms of a geometric sequence: $S_3 = t_1 + t_2 + t_3$
For the sequence above, $S_3 = 1 + 2 + 4 = 7$

S_4 - is the sum of the first 4 terms of a geometric sequence: $S_4 = t_1 + t_2 + t_3 + t_4$
For the chessboard sequence, $S_4 = 1 + 2 + 4 + 8 = 15$

For this question, we need to know the value of S_{64}

Here's a way to find the value of a sum of *n* terms for any geometric series, S_n

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cancel{\ldots} + \cancel{ar^{n-2}} + \cancel{ar^{n-1}}$$
$$-rS_n = \quad \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cancel{\ldots} + \cancel{ar^{n-2}} + \cancel{ar^{n-1}} + ar^n$$

$S_n - rS_n = a - ar^n$

Subtracting, we get: $S_n - rS_n = a - ar^n$

Factor: $S_n(1-r) = a(1-r^n)$

Divide: $S_n = \dfrac{a(1-r^n)}{1-r}$

$S_n = \dfrac{a(1-r^n)}{1-r}$

$1 + 2 + 4 + \ldots$

$r = 2$

$S_{15} = \dfrac{1(1-2^{15})}{1-2}$

$= \dfrac{(1-2^{15})}{-1} = \dfrac{-32767}{-1}$

$= \boxed{32767}$

$S_{64} = \dfrac{1(1-2^{64})}{1-2} \doteq 1.844 \times 10^{19}$ (quintillion!)

$= 18,446,744,073,709,551,615$ grains of wheat!!

For any geometric sequence, the sum of the first n terms is found using:

$$S_n = \frac{a(1-r^n)}{1-r}$$

a = first term
r = common ratio, $r \neq 1$
n = number of terms to add up

Another form of the formula: $S_n = \dfrac{a - lr}{1-r}$ $\qquad l$ = the last term

$a = 4 \qquad r = \frac{12}{4} = 3 \qquad n = 9$

To Try:
1. Given the geometric series: 4 + 12 + 36 + . . . find the sum of the first 9 terms.

To Try:

$a=4 \quad r=\frac{12}{4}=3 \quad n=9$

1. Given the geometric series: 4 + 12 + 36 + . . . find the sum of the first 9 terms.

$$S_9=\frac{4(1-3^9)}{1-3}=\boxed{39\,364}$$

2. The sum of the first 12 terms of a geometric series is $\underbrace{24\,570}_{S_{12}}$. The common ratio is -2 $=r$.
What is the first term? $=a$

$n=12$

$$S_n=\frac{a(1-r^n)}{1-r}$$

$$24\,570=\frac{a(1-(-2)^{12})}{1-(-2)}$$

$$24\,570=\frac{a(1-4096)}{3}$$

$$3\times 24\,570=\frac{a(-4095)}{\cancel{3}}\times\cancel{3}$$

$$\frac{73\,710}{-4095}=\frac{-\cancel{4095}a}{-\cancel{4095}}$$

$$\boxed{a=-18}$$

3a) Find the sum of this series. 0.025 + 0.075 + . . . + 54.675

$l=$ last term

$$S_n=\frac{a-lr}{1-r}$$

$$r=\frac{0.075}{0.025}=3$$

$$S_n=\frac{0.025-(54.675)(3)}{1-3}$$

$$\boxed{S_n=82}$$

b) Use an algebraic method to determine how many terms there are in the above sum.

$$t_n=ar^{n-1}$$

$$\frac{54.675}{0.025}=\frac{\cancel{0.025}(3)^{n-1}}{\cancel{0.025}}$$

$$2187=3^{n-1}$$

$$3^7=3^{n-1}$$

$$\Rightarrow 7=n-1$$

$$\boxed{n=8}$$

4. An oil well produced 10 000 barrels of oil in the first month of production. Each month production is reduced by 10%. What is the total amount of barrels of oil produced by this oil well in its first two years of production?

$10\,000,\ 10\,000(0.9),\ \ldots$

$r=0.9$

$$S_n=\frac{a(1-r^n)}{1-r}$$

$$S_{24}=\frac{10\,000(1-0.9^{24})}{1-0.9}$$

$= 92\,023$ barrels

G Practice

C_25 More Sequences and Series Practice

Sequences and Series – more practice (14 questions)

1. Is the following sequence geometric?

a) 10, 15, 22.5, 33.75,...

b) 7, 14, 21, 28,...

2. Find the common ratio, r, of each geometric sequence

a) $-1, -5, -25, -125, \ldots$

b) $-200, 100, -50, -25, \ldots$

3. Find the next three terms of the following sequence

a) 386561, 55223, 7889, ____, ____, ____

b) $-\frac{1}{5}, -\frac{1}{15}, -\frac{1}{45}$, ____, ____, ____

4. Find a formula for the nth term of each geometric sequence.

a) $a = 4,\ t_{13} = 16384$

b) $t_3 = 5,\ t_6 = 135$

5. The seventh term of a geometric sequence is 1215 and the fourth term is 45. Find the common ratio, then find the value of the ninth term.

6. A population of rabbits is growing at a rate of 8% a year. If there are 160 rabbits in the initial population, create a general term equation, t_n, describing this sequence. Use it to find the number of rabbits after 6 years.

7. Find the sum of the following geometric series. If necessary, round to 2 decimal places.

a) $729 - 243 + 81 - 27 + \ldots$ (10 terms)

b) $7 + 14 + 28 + 56 + \ldots + 7168$

c) $\sum_{n=4}^{10} 5(2)^n$

8. Find the common ratio of a geometric series with a first term of 38 and a sum to infinity of 76.

9. Find the general term, t_n , for the described sequences:

a) geometric, beginning: $-2, 1, -\frac{1}{2}, \frac{1}{4},..$

b) geometric, with $t_3 = 75$ and $r = 5$

c) geometric, with $t_4 = 5$ and $r = \frac{1}{4}$

10. Find the 25th term of the following geometric sequence: $2, 2\sqrt{3}, 6,...$

11. List the first five terms of the geometric sequence with $t_3 = 8$ and $r = -\frac{1}{2}$.

12. Find the requested sum for each geometric sequence.

a) Find S_{12} correct to 2 decimal places, for $a = 5$, $r = \frac{2}{3}$

b) Find S_9 for $a = -3$ and $r = 2$

c) Find the sum of the first 11 terms of the geometric series that begins $7 - 14 + 28 - ...$

13. Determine the sum, if possible:

a) $\sum_{i=1}^{\infty} -4\left(\frac{4}{5}\right)^i$

b) $\sum_{i=1}^{6} 2(3)^i$

c) $\sum_{i=1}^{\infty} 5\left(\frac{4}{3}\right)^i$

d) $\sum_{i=1}^{\infty} 5\left(\frac{2}{3}\right)^i$

14. A helium balloon rises 80 meters the first minute after it is released. Each minute after that it rises 15% less than the previous minute. How high does the balloon rise in total?