Class_25 Dec 6 - Infinite Series and Sigma Notation

Sunday, November 20, 2022 3:24 PM

Tonight's Class:

- **Infinite Series (G.3)**
- **Sigma Notation (G.4)**
- **Unit 4 Test next class, Thursday, Dec 8**

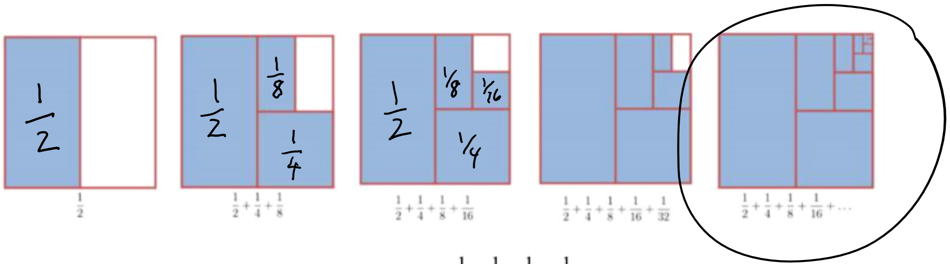
Please:

1. **Hand in the Chapter 9 Hand-in. Please make sure your name is on it.**
2. **Clear your desk of any materials except for your calculator & something to write with.**
3. **While other people are still finishing, respect them by being quiet. You can leave the classroom if you wish, but be back in time for the rest of class.**

Reminder

Coming up - "Ugly Xmas Sweater Days" at LEC

G. 3 Infinite Geometric Series



Consider the series illustrated above: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots$

- What do you notice about the terms? getting smaller
- What do you think the sum is? 1

An ***infinite geometric series*** is one with an infinite number of terms – there is no last term.

With an infinite series, we will find that either:

Successive terms get smaller and smaller and smaller.

- The common ratio, *r*, has a value between -1 and 1. ☆ $-1 < r < 1$
- This type of series is called ***convergent.***
- We **CAN** find the sum by using the formula: $S = \frac{a}{1-r}$

Successive terms stay the same size, or continue to get larger and larger

- The common ratio, *r*, satisfies either $r \geq 1$ *or* $r \leq -1$.
- This type of series is called ***divergent***.
- We **CANNOT** find the sum of this series. We say: "No finite sum."

A geometric series has a **finite sum**, or **converges**, only when $-1 < r < 1$

$$S = \frac{a}{1-r}$$, **only true if -1 < *r* < 1**

If you have an infinite geometric series where $r \geq 1$ or $r \leq -1$, it will <u>not</u> have a finite sum, it will **diverge.**

$\boxed{S=\frac{a}{1-r}}$

1) find r
2) decide if there's a sum, & what it is

$-1<r<1$

If r is not between −1 and 1, we say "No finite sum"

To Try:

1. Determine whether each infinite geometric series has a finite sum. If it does have a finite sum, find its value.

a) $10-\frac{20}{3}+\frac{40}{9}-\ldots$

$r=\frac{\frac{-20}{3}}{10}=\frac{-20}{3}\cdot\frac{1}{10}=\frac{-20}{30}=-\frac{2}{3}$

Since $-1<-\frac{2}{3}<1$, it has a finite sum

$S=\frac{10}{1-(-\frac{2}{3})}=\frac{10}{\frac{3}{3}+\frac{2}{3}}=\frac{10}{\frac{5}{3}}=10\cdot\frac{3}{5}=\frac{30}{5}=\boxed{6}$

b) $0.1+0.3+0.9+\ldots$

$r=\frac{0.3}{0.1}$

$r=3$

No finite sum!

c) $32+8+2+0.5+\ldots$

$r=\frac{8}{32}$

$r=\frac{1}{4}$

$-1<\frac{1}{4}<1$, so yes we can use the formula.

$S=\frac{32}{1-\frac{1}{4}}=\frac{32}{\frac{3}{4}}=\frac{32}{1}\cdot\frac{4}{3}=\boxed{\frac{128}{3}\text{ or }42\frac{1}{3}}$

d) $-1-\frac{3}{4}-\frac{9}{16}-\frac{27}{64}\ldots$

$r=\frac{-\frac{3}{4}}{-1}$

$r=\frac{3}{4}$

$S=\frac{-1}{1-\frac{3}{4}}=\frac{-1}{\frac{4}{4}-\frac{3}{4}}=\frac{-1}{\frac{1}{4}}=-1\cdot 4=\boxed{-4}$

2a) The sum of an infinite geometric series is 63 and the first term is 21. Find the common ratio. $\rightarrow r=?$

$S=63$, $a=21$

$S=\frac{a}{1-r}$

$(1-r)(63)=\left(\frac{21}{\cancel{1-r}}\right)\times\cancel{(1-r)}$

$(1-r)(63)=21$

$63-63r=21$
$-63\qquad -63$

$\frac{-63r}{-63}=\frac{-42}{-63}$

$r=\frac{42}{63}=\frac{6}{9}=\boxed{\frac{2}{3}}$

b) The sum of an infinite geometric series is 24/7 and $r=-\frac{3}{4}$. Find the first term, a.

$\frac{24}{7}=\frac{a}{1-(-\frac{3}{4})}$

$\frac{24}{7}=\frac{a}{1+\frac{3}{4}}$

$\frac{24}{7}=\frac{a}{\frac{4}{4}+\frac{3}{4}}$

$\frac{7}{4}\left(\frac{24}{7}\right)=\left(\frac{a}{\cancel{\frac{7}{4}}}\right)\cancel{\frac{7}{4}}$

$\frac{7}{4}\cdot\frac{24}{7}=a$

$\frac{24}{4}=a$, $\boxed{a=6}$

c) An infinite geometric series has the terms $a=32$ and $t_4=4$. Determine the sum of the infinite series.

$\boxed{t_n=ar^{n-1}}$

$n=4$

$t_4=4=32r^{4-1}$

$\frac{4}{4}=\frac{32r^3}{4}$

$\frac{1}{8}=\frac{8r^3}{8}$

$r^3=\frac{1}{8}$

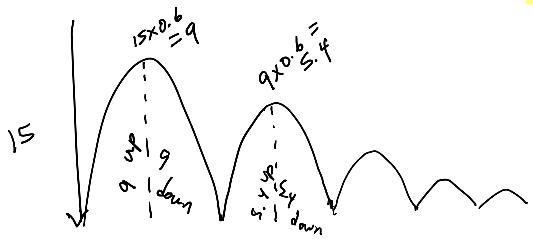
$r=\frac{1}{2}$

$S=\frac{a}{1-r}$

$S=\frac{32}{1-\frac{1}{2}}=\frac{32}{\frac{1}{2}}=32\cdot\frac{2}{1}=\boxed{64}$

4. A ball is dropped from a height of 15 meters and bounces to 60% of the previous height. How far has the ball traveled when it ***comes to rest***?

(infinite)



$S=\frac{a}{1-r}$

$=\frac{15}{1-0.6}=\frac{15}{0.4}=37.5$

$= $ all the DOWNs

We need "UPs" also, let's multiply by 2:

$37.5\times 2=75$

must subtract off the extra 15 : $75-15=\boxed{60\text{ m}}$

Σ = upper-case sigma

σ = lower-case sigma

G. 4 Sigma Notation

Series are sums. We have been writing them in an expanded form, like this example:

$$32+8+2+0.5$$

Here is a different way to write the same series: $\displaystyle\sum_{n=3}^{6} 32(0.25)^{n-3}$

This notation, ***sigma notation,*** uses the upper-case Greek letter sigma, Σ, and means we need to "SUM UP" all the terms in the series generated by the expression after the sigma. Sigma notation is used in many mathematics and science resources, so it's useful to understand how it works.

To ***EXPAND*** a series written in sigma notation

- Substitute the bottom number – the one written below the sigma – into the expression. This gives the first term of the series.
- Keep substituting in consecutive numbers to get the next terms of the series.
- The last number to substitute in is the one written above the sigma. This gives the last term of the series.

n=3

$$\sum_{n=3}^{6} 32(0.25)^{n-3} = 32(0.25)^{3-3} + 32(0.25)^{4-3} + 32(0.25)^{5-3} + 32(0.25)^{6-3}$$

$$= 32(0.25)^0 + 32(0.25)^1 + 32(0.25)^2 + 32(0.25)^3$$

$$= 32 + 8 + 2 + \frac{1}{2} = 42.5$$

evaluate

To Try:

1. Expand and evaluate: $\displaystyle\sum_{n=2}^{4} 3\left(\frac{1}{2}\right)^{n-1} = 3\left(\tfrac{1}{2}\right)^{2-1} + 3\left(\tfrac{1}{2}\right)^{3-1} + 3\left(\tfrac{1}{2}\right)^{4-1}$

$$= 3\left(\tfrac{1}{2}\right)^1 + 3\left(\tfrac{1}{2}\right)^2 + 3\left(\tfrac{1}{2}\right)^3$$

$$= \frac{3}{2}\cdot\frac{4}{4} + \frac{3}{4}\cdot\frac{2}{2} + \frac{3}{8}$$

$$= \frac{12}{8} + \frac{6}{8} + \frac{3}{8} = \boxed{\frac{21}{8} \text{ or } 2.625}$$

(top # – bottom #) + 1 = how many terms there are

2. Evaluate: $\displaystyle\sum_{k=5}^{21} 4(2)^{k-2}$

$$= 4(2)^{5-2} + 4(2)^{6-2} + \cdots + 4(2)^{21-2}$$

$$= 4(2)^3 + 4(2)^4 + \cdots$$

$$= \underbrace{32 + 64 + \cdots}_{\text{how many do we need to add up?}}$$

17 terms

$a = 32$

$r = \frac{64}{32} = 2$

$n = 17$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{32(1-2^{17})}{1-2}$$

$$= \boxed{4,194,272}$$

To ***WRITE*** a geometric series in sigma notation

- Figure out the values of ***a*** and ***r***
- Write a general expression for the series, using the format: $t_n = ar^{n-1}$
- Determine how many terms you have, ***n***. Use this for the number above the sigma.

Example

Write the given series in sigma notation: $5+15+45+135+405+1215+3645$ $n=7$

$a=5$ $r=\frac{15}{5}$ $r=3$

$$\boxed{\sum_{n=1}^{7} 5(3)^{n-1}}$$

To Try

1a) Write the series: $3 + 9 + 27 + 81 + 243$ in sigma notation.

$t_n = ar^{n-1}$

$$\sum_{n=1}^{5} 3(3)^{n-1}$$

$t_n = ar^{n-1}$

b) Write the series: $\underbrace{20 + 40 + 80 + \ldots + 163\,840}_{\text{how many terms?}}$ in sigma notation.

$\frac{163\,840}{20} = \frac{\cancel{20}(2)^{n-1}}{\cancel{20}}$

$8192 = 2^{n-1}$

$2^{13} = 2^{n-1}$

$\Rightarrow 13 = n-1$

$n = 14$

$$\sum_{n=1}^{14} 20(2)^{n-1}$$

Expand it, the first 2 terms.

2. Determine each sum.

a) $\sum_{k=4}^{12} 8\left(\frac{1}{2}\right)^{k-2}$ $= 8\left(\frac{1}{2}\right)^{4-2} + 8\left(\frac{1}{2}\right)^{5-2} + \ldots$

$= 8\left(\frac{1}{2}\right)^{2} + 8\left(\frac{1}{2}\right)^{3} + \ldots$

$= 8\left(\frac{1}{4}\right) + 8\left(\frac{1}{8}\right) + \ldots$

$= 2 + 1 + \ldots$

$a = 2$

$r = \frac{1}{2}$

$n = (12-4)+1 = 8+1 = 9$

$S_n = \frac{a(1-r^n)}{1-r}$

$= \frac{2(1-(1/2)^9)}{1-1/2} = 3.9921875$

b) $\sum_{n=7}^{17}(2)^n$ $= 2^7 + 2^8 + \ldots$

$= 128 + 256 + \ldots$

$a = 128$

$r = \frac{256}{128}$

$r = 2$

$n = (17-7)+1$

$n = 11$

$S_n = \frac{a(1-r^n)}{1-r}$

$S_{11} = \frac{128(1-2^{11})}{1-2}$

$= \boxed{262\ 016}$

2. Find the sum of these series, correct to two decimal places.

a) $\sum_{k=1}^{\infty}100(0.3)^{k-1}$ $= 100(0.3)^{1-1} + 100(0.3)^{2-1} + \ldots$

$= 100 + 30 + \ldots$

infinity

$S = \frac{a}{1-r}$

$a = 100$

$r = \frac{30}{100} = 0.3$

$S = \frac{100}{(1-0.3)}$

$\doteq 142.86$

b) $\sum_{k=1}^{\infty}\frac{4}{5}\left(\frac{2}{3}\right)^{n-1}$ $= \frac{4}{5}\left(\frac{2}{3}\right)^{1-1} + \frac{4}{5}\left(\frac{2}{3}\right)^{2-1} + \ldots$

$= \frac{4}{5} + \frac{8}{15} + \ldots$

$a = 4/5$

$r = \frac{8}{15} \div \frac{4}{5}$

$8/15 \cdot \frac{5}{4} = \frac{2}{3}$

$S = \frac{4/5}{1-2/3}$

$= \frac{12}{5} = \boxed{2.4}$

Worksheets
More Sequences & Series Practice
Unit 4 Practice Test

Thursday, Dec 8
- **Unit 4 Test**
- **Hand in assignment on Geometric Sequences & Series**

- **Last day to hand in any assignments**

  <u>Tuesday, Dec 13</u>
  Optional re-write day