Chapter 1: Function Transformations
1.0 Review

A ***relation*** is a set of ordered pairs $(x,\ y)$.

For example: $\{(-1, 6)\ (2, 8)\ (5, 10)\ (8, 12)\}$

A ***function*** is a special type of relation.

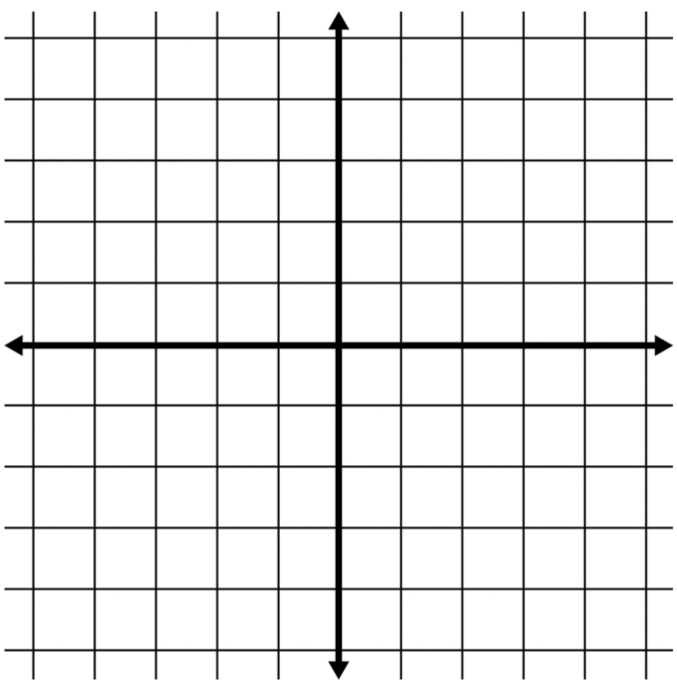
- A function is like a machine. For each x-value, the function follows a rule to create exactly ONE y-value that goes with that x-value.
- **Vertical Line Test:** Function graphs contain NO points that are directly above one another.

  For example, a function CANNOT contain both $(2,8)$ and $(2,5)$

Graphing

Ordered pairs are graphed on a coordinate system: $(x,\ y)$

x*-coordinates** tell how far to move left or right from the ***origin, $(0,0)$
***y*-coordinates** tell how far to move up or down from the origin.



Plotting points.

Plotting graphs,

Function Notation

$y = f(x)$

- means y is a function of x, so the y-value depends on the x-value we choose
- is read " y equals f of x "

To Try
1) Given the function $f(x) = 3 - 4x$, find the value of $f(-4)$

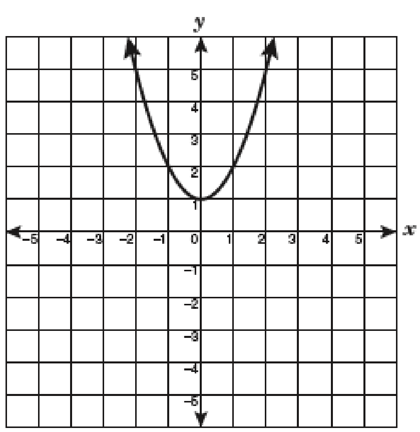
2) Given the function $g(x) = -2x^2 + 5$, find the value of $g(2)$

Domain

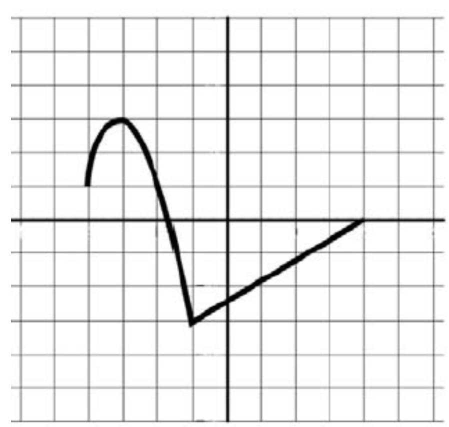
Range

Find the domain and range for each graph below, and write it using set notation.

a)



b)

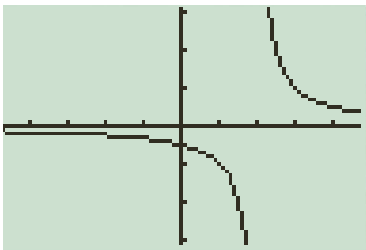


<u>To Try</u>

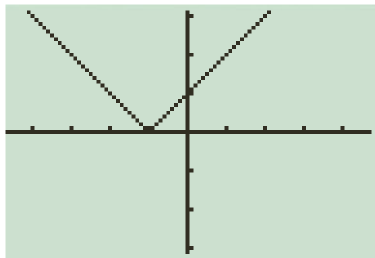
We can use technology to create the graph of a function.

- Use a graphing calculator to graph the following functions. Your graphs should match the graphs shown below.
- Determine the ***domain*** and ***range*** for each one.

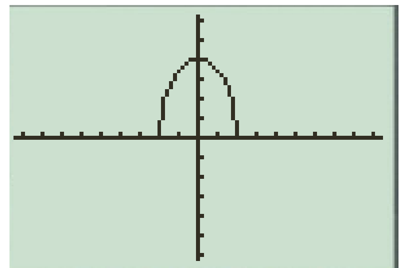
a) $f(x) = \dfrac{1}{x-2}$



b) $f(x) = |x+1|$



c) $f(x) = \sqrt{16-4x^2}$



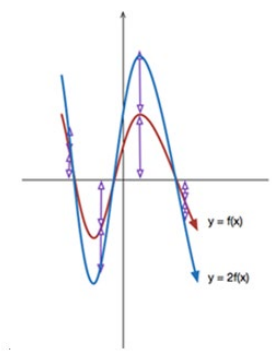
Used [-9.4, 9.4] [-6.2, 6.2]

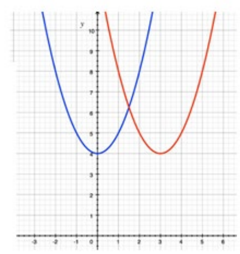
Transformations

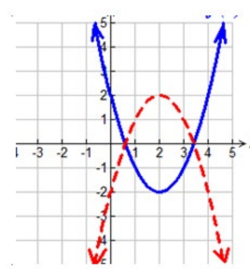
We often sketch the graphs of functions. If we change a function's equation, the new equation produces a new, TRANSFORMED, graph.

Transformations include:

translations
reflections
stretches







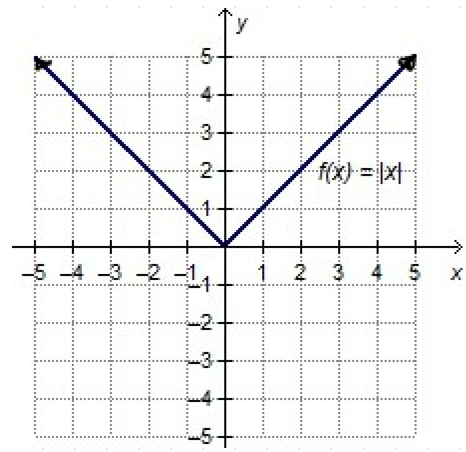
When a graph is transformed, each point on the graph is affected by the transformation.

Suppose that the x-coordinates for all the points on a graph are increased by three units. Here is a way to show how the points are changed.

MAPPING $(x, y) \rightarrow (\_\_\_\_, \_\_\_\_)$

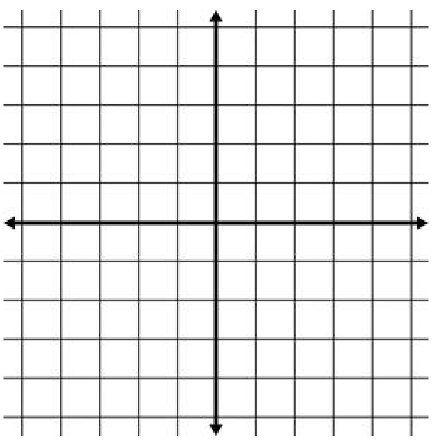
1.1 Horizontal and Vertical Translations

The graph of the base absolute value function is shown at right, and below are three transformed equations.

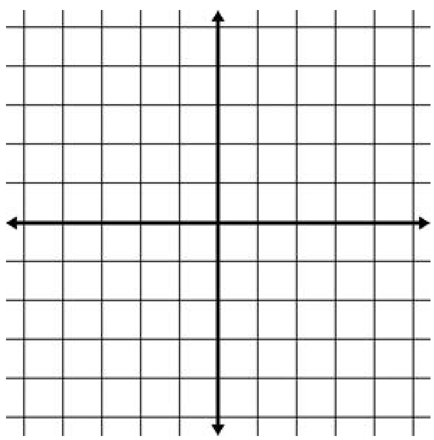


For each one:
- Sketch its graph on the grid.
- Describe, in words, the transformation that happened.
- Describe the transformation by giving its ***mapping***.
- State the domain and range.

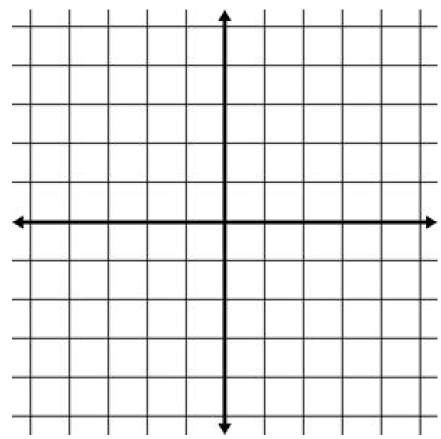
$f(x) = |x-4|$



$f(x) = |x|+3$



$f(x) = |x+2|-4$



Points on an original graph correspond with points on a transformed graph, often called the ***image graph***. We say that each original point is ***mapped*** to an ***image point***.

Horizontal translation

$$y - k = f(x - h)$$

Vertical translation

Often equations are arranged with the "*y*" term isolated:

Horizontal translation

$$y = f(x - h) + k$$

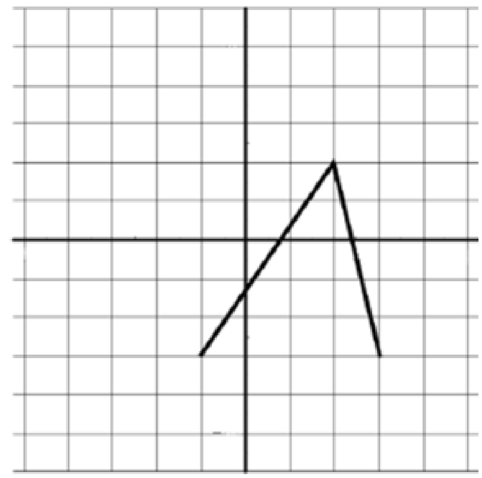
Vertical translation

TRANSLATIONS – sliding graphs left/right/up/down

Some specific examples:
- when x is replaced with $x-8$, the graph will move 8 right.
- when x is replaced with $x+6$, the graph will move 6 left.
- when y is replaced with $y-4$, the graph will move 4 up.
- when y is replaced with $y+7$, the graph will move 7 down.

Base Function Equation	Transformed Equation	Mapping	Point on original graph	Its image point
$y=x^2$	$y-4=x^2$		$(-3, 9)$	
$y=x+5$	$y=(x-3)+5$		$(2,7)$	
$y=\log_5 x$	$y=\log_5(x-2)+3$		$(25, 2)$	
$y=2^x$	$y=2^{x-3}+8$		$\left(-1, \frac{1}{2}\right)$	
$y=\frac{2}{x-4}$	$y=\frac{2}{(x+3)-4}+6$		$\left(8, \frac{1}{2}\right)$	
$x^2+y^2=16$	$(x-5)^2+(y+3)^2=16$		$(-4, 0)$	

To Try
Shown is the graph of $y = f(x)$.



a) Identify the transformations that result when the equation is changed to: $y - 2 = f(x+3)$

b) Make a table of key points on the original graph and the corresponding image points on the image graph.

Base

x	y

Image

c) Sketch the image graph.

d) State the domain and range of the image graph. (Assume that the line segments stop.)

Example
Given the mapping notation for a transformation, we can write the transformed equation.

a) Mapping notation $(x,\ y) \rightarrow (x-8,\ y+3)$
Original function $y = f(x)$
New function

b) Mapping notation $(x,\ y) \rightarrow (x+4,\ y-9)$
Original function $y = f(x)$
New function

1.2 Reflections and Stretches

Reflections

Across the x-axis

Original key points

x	y

Reflected key points

x	y

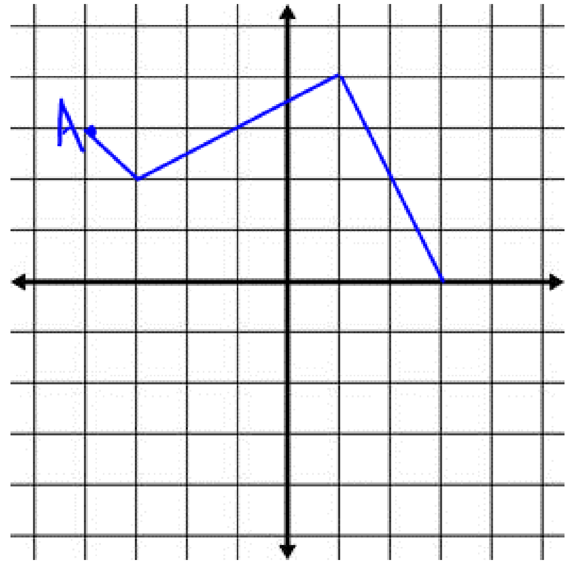


Image point for point A:
Original equation: $y = f(x)$
New equation:
Mapping:

Across the y-axis

Original key points

x	y

Reflected key points

x	y

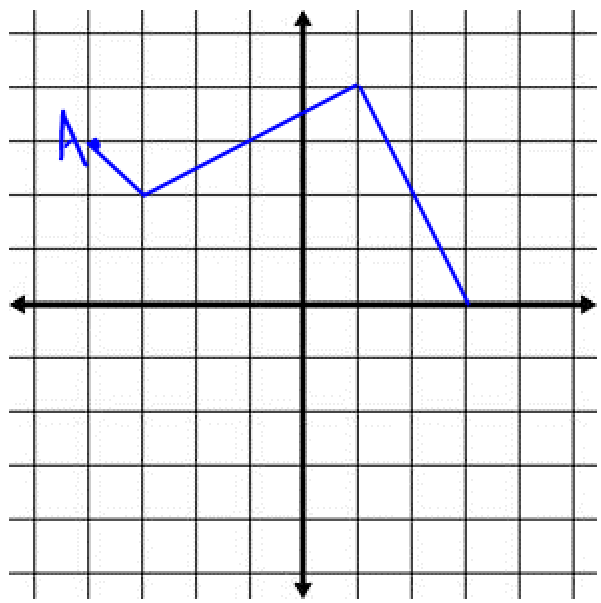


Image point for point A:
Original equation: $y = f(x)$
New equation:
Mapping:

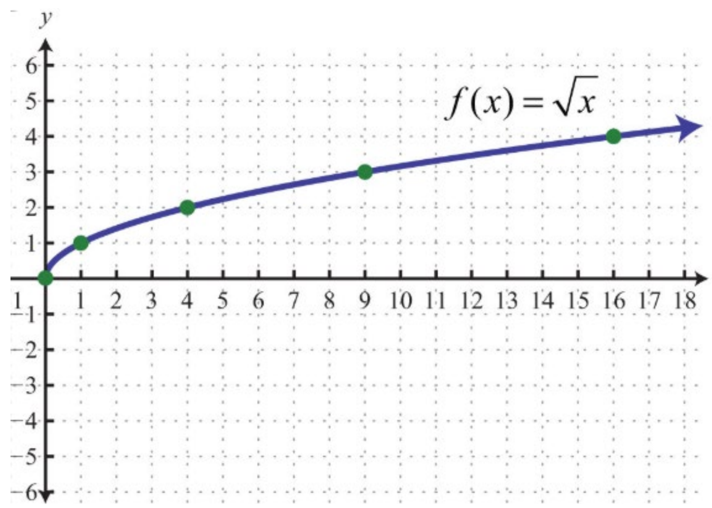
Points that do not change under a given transformation are called ***invariant points***.
Which points are invariant in the reflections above?

REFLECTIONS – reflecting graph across either *y*-axis or *x*-axis

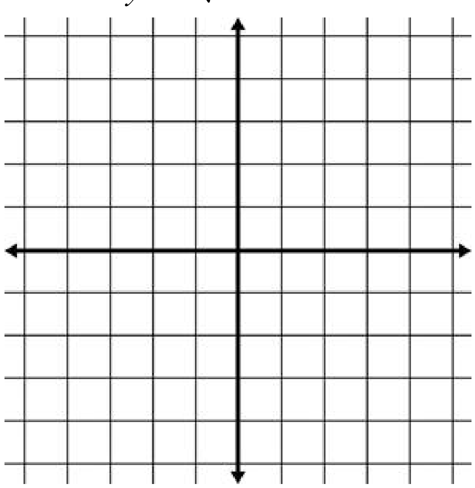
Some specific examples:
- when x is replaced with $-x$, the graph will be reflected across the y-axis.
- when y is replaced with $-y$, the graph will be reflected across the x-axis.
- If instead of $y = f(x)$ we have $y = -f(x)$, the graph is reflected across __________

The graph of the base radical function is shown.

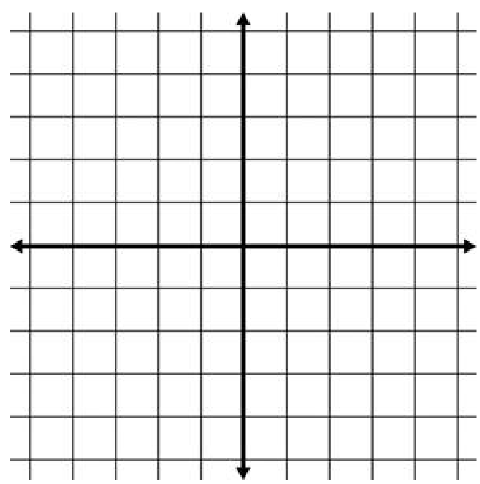
For each transformed equation below
- Sketch its graph on the grid.
- Give its domain and range, using set notation.
- Describe, in words, what change occurred.
- Describe the transformation by giving its ***mapping***.



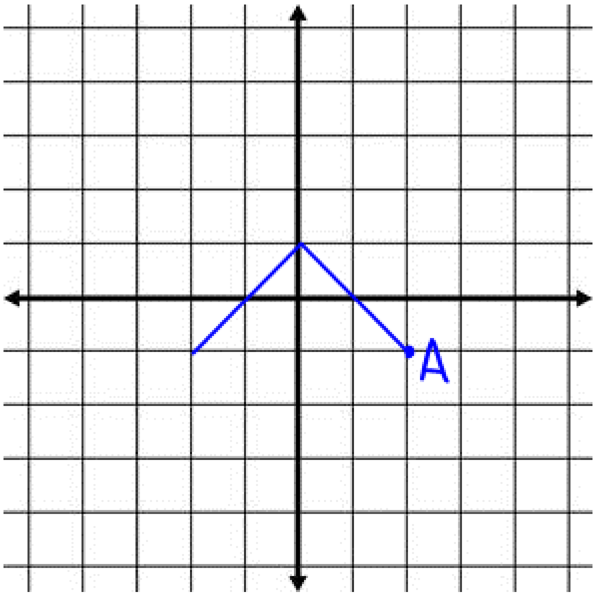
$y = -\sqrt{x}$



$y = \sqrt{-x}$



Stretches



Vertical – all y-values are multiplied by a number, the stretch factor

Key points

x	y

Image points

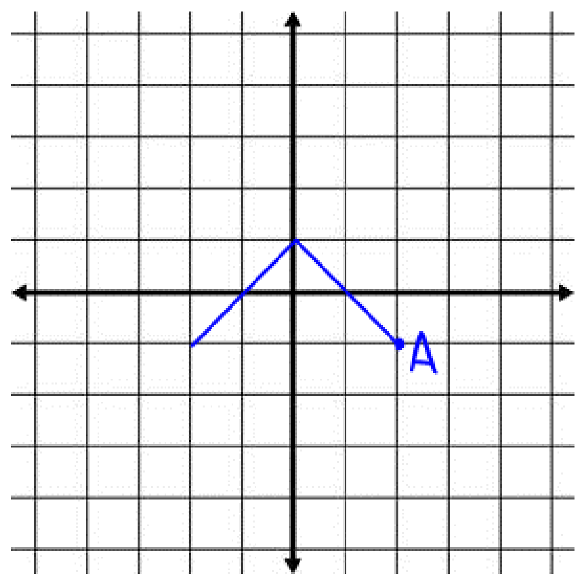
Mapping:

Horizontal – all x-values are multiplied by a number, the stretch factor

Key points

x	y

Image points

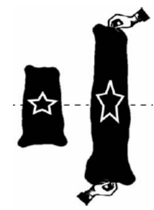


Mapping:

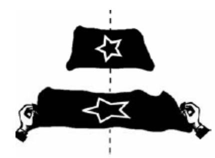
Which points are invariant in the stretches above?

STRETCHES – horizontal and vertical stretches

When $y = f(x)$ is changed to $y = a\,f(x)$, each point on the original graph has its y-value multiplied by "a."
This is a ***vertical stretch***, by a factor of a.



When $y = f(x)$ is changed to $y = f(bx)$, each point on the original graph has its x-value multiplied by the <u>*reciprocal*</u> of b. This is a ***horizontal stretch*** by a factor of $\frac{1}{b}$.



When the stretch factor is a number between -1 and 1, we call it a ***compression***. Otherwise, we call it an ***expansion.***

Examples

a) Identify each change, when $y = f(x)$ is changed to:

$y = 8f(x)$ $\qquad y = f(2x)$ $\qquad y = \frac{1}{2}f(x)$

$y = f\left(\frac{1}{4}x\right)$ $\qquad 4y = f(x)$ $\qquad \frac{1}{2}y = f(x)$

b) Write the new equation that causes $y = f(x)$ to be stretched as follows:

Vertical stretch, by $\frac{2}{3}$ $\qquad$ Horizontal stretch, by $\frac{5}{2}$

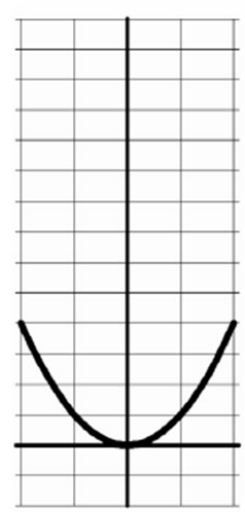
<u>To Try</u>

The graph of $y = f(x)$ is shown at right. When changed to $y = 3f(x)$,

- identify the transformation
- complete the table and mapping
- sketch the graph of $y = 3f(x)$

x	y
-2	4
-1	1
0	0
1	1
2	4

Image points



$(x, y) \rightarrow$

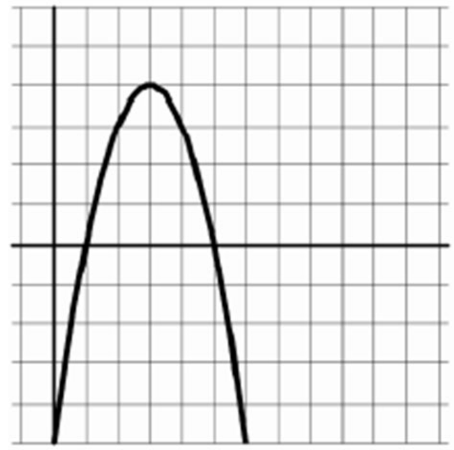
To Try

The graph of $y = f(x)$ is shown at right. When changed to $y = f\left(\frac{1}{2}x\right)$,

- identify the transformation
- complete the table and mapping
- sketch the graph of $y = f\left(\frac{1}{2}x\right)$

x	*y*
0	-5
1	0
3	4
5	0
6	-5

Image points



$(x, y) \rightarrow$

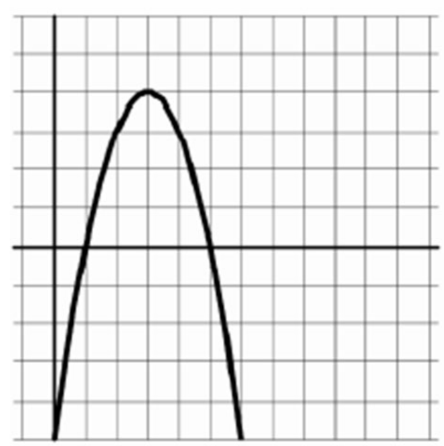
To Try

The graph of $y = f(x)$ is shown at right. When changed to $y = -\frac{1}{2}f(x)$,

- identify the transformation
- complete the table and mapping
- sketch the graph of $y = -\frac{1}{2}f(x)$

x	*y*
0	-5
1	0
3	4
5	0
6	-5

Image points



$(x, y) \rightarrow$

1.3 Combining Transformations

Summary of Transformations. Original Equation, $y = f(x)$

Translations

	Graph moves…	Mapping
$y+4=f(x)$		$(x,\ y) \rightarrow$
$y-5=f(x)$		
$y=f(x+2)$		
$y=f(x-6)$		

Stretches

	Graph is stretched…	Mapping
$y=5f(x)$		
$\frac{3}{2}y=f(x)$		
$y=f(4x)$		
$y=f\left(\frac{1}{3}x\right)$		

Reflections

	Reflects across…	Mapping
$y=-f(x)$		
$y=f(-x)$		

Vertical stretch, factor *a*

Horizontal translation

$$y = a\,f\,(b(x-h)) + k$$

Horizontal stretch, factor *1/b*

Vertical translation

Question…….

If more than one transformation is applied to a graph, does the ***order*** in which the transformations are done change the final graph?

$

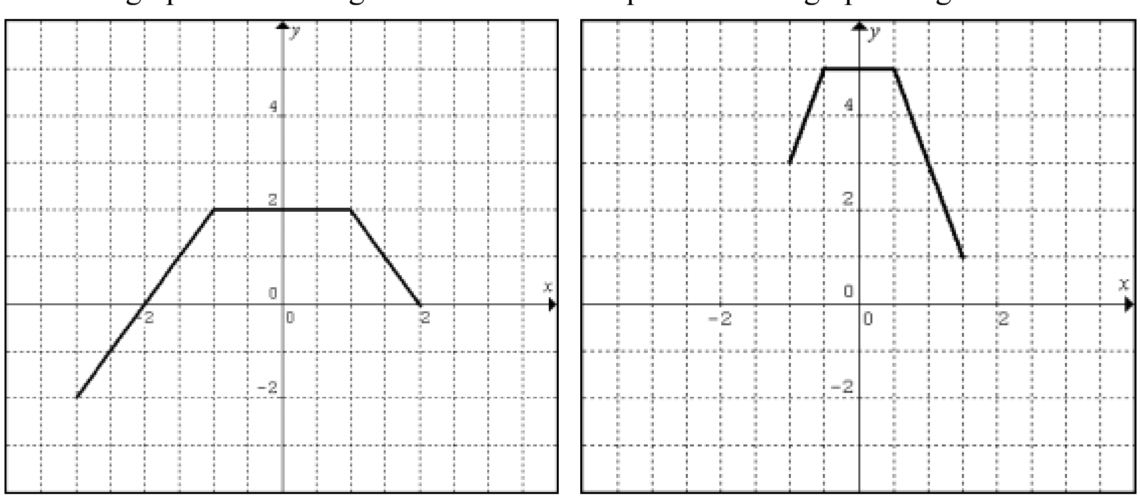
Apply transformations in this order, to get the final graph:

Example List all the transformations, then give the mapping.

a) $y = -4f\left(\frac{1}{2}(x-3)\right)+6$

b) $y = 2f(3x-6)+5$

Example
Identify the transformations that need to happen, to change the graph of $y = f(x)$ on the left to the graph shown at right. Determine the equation of the graph at right.



Combining Transformations – Radicals

The changes we have discussed work with any function. Here are some questions to try, relating to the base radical function $y = \sqrt{x}$.

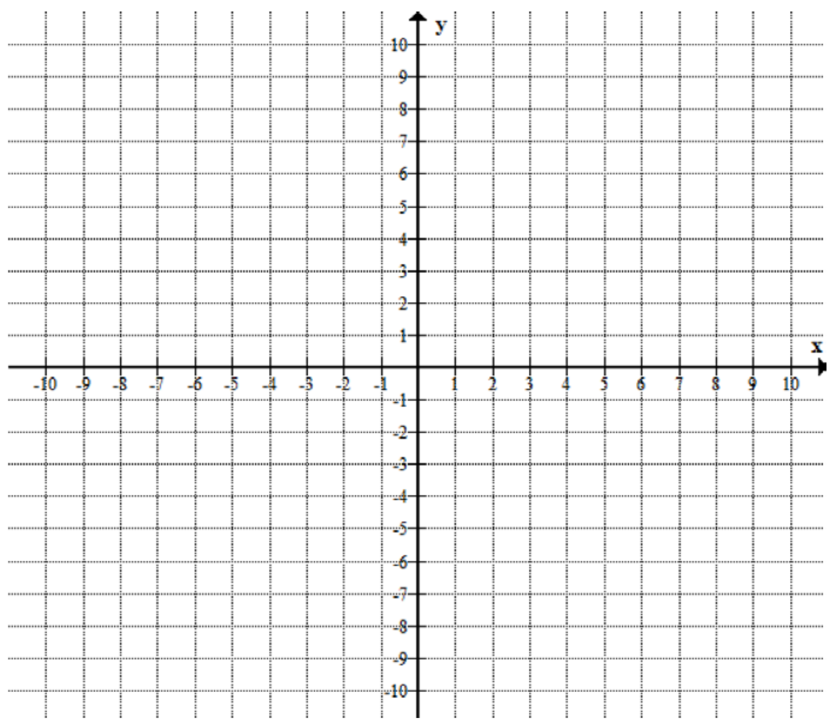
1) List the transformations that occur when the base radical function is changed to:
$y + 9 = \frac{1}{2}\sqrt{-3x - 12}$. Give the mapping.

2) Given the mapping, which acts on the base radical function, write the new, transformed equation.

$$(x, y) \rightarrow \left(5x + 2, \frac{1}{2}y - 4\right)$$

3) Complete the table of values for the base function, $y = \sqrt{x}$, and for the transformed equation, $y = -2\sqrt{x + 4} - 3$. Give the mapping, and the final graph's domain and range.

x	y



domain:

range:

mapping:

1.4 Inverses

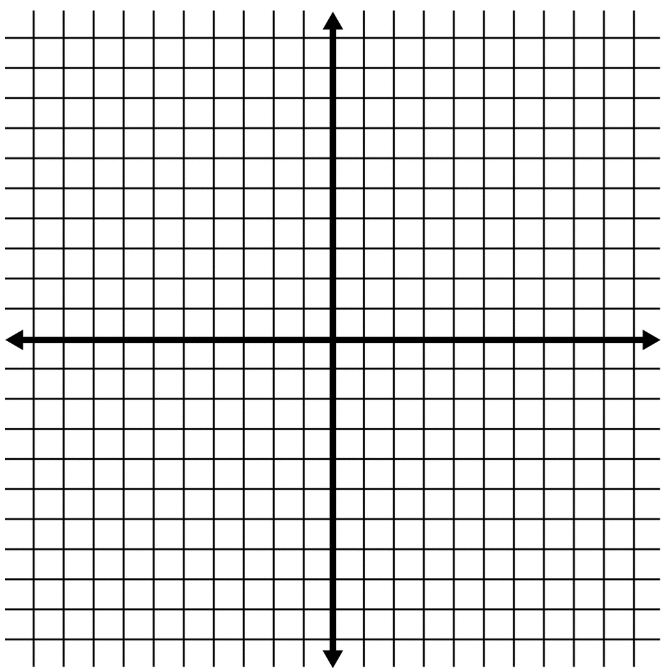
Inverse Operations

The operation that reverses the effect of another operation.
Examples of inverse operations?

Inverse of a Relation

The inverse of a relation is found by interchanging the x-coordinates and y-coordinates of each of the ordered pairs in the relation. Each ordered pair in the relation, (x, y), is changed to the ordered pair (y, x) to form a point on the inverse of the relation.

Example For the relation $\{(0,0)\ (1,1)\ (2,4)\ (3,9)\}$, what ordered pairs form its inverse?



Plot the original points and connect them.

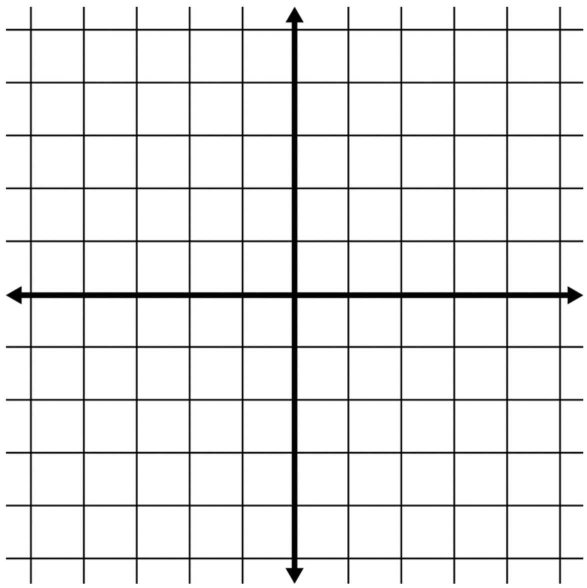
Do the same for the inverse points.

What do you notice about the two graphs?

Example

a) Complete the table below for the equation $f(x) = 2x + 3$. Plot the points on the grid and connect them with a line segment.

b) Complete the table for the ***inverse*** of $f(x)$. Plot these new points on the same grid and connect them with a line segment.



function

$y = f(x)$ $y = 2x + 3$	
x	***y***
–4	
–3	
–1	
0	

inverse

$x = f(y)$	
x	***y***

c) Use the graphs to complete the table below.

	Original function	***Inverse***
domain		
range		
x-intercept		
y-intercept		

The graph of a relation and its inverse are always reflections of each other across the line $y = x$.

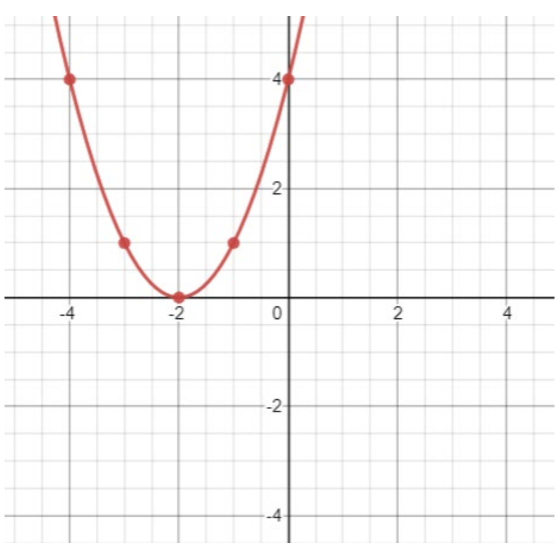
Example Find the equation of the inverse for the function on the previous page, $f(x) = 2x + 3$.

Example Find the equation of the inverse for $f(x) = (2x - 1)^2 + 4$.

Example Find the equation of the inverse for $f(x)=\sqrt{4x-5}$.

Example Find the equation of the inverse for $f(x)=\dfrac{8x+1}{2x-5}$.

Example Shown is the graph of $f(x)=(x+2)^2$.



a) Graph the inverse of $f(x)$ on the same grid.

b) How can we restrict the domain of $f(x)$ so that the inverse graph is a function?

Chapter 3: Polynomial Functions

3.1 Characteristics of Polynomial Functions

- ***term*** - a single number (called a constant), a variable, or numbers and variables multiplied together

- a polynomial can just one term, or it can be made up of several terms added/subtracted together

- ***exponents*** of polynomial terms must be positive integers (no negative, fractional or decimal exponents)

- ***coefficients*** of polynomial terms must be real numbers (no square-roots of negative numbers). In this chapter, we will only use coefficients that are integers.

- the ***degree of a constant*** is zero, ***degree*** of other terms is the exponent of x term

- the ***degree of a polynomial*** is found by looking at the degree of each of term and choosing the largest one

- the ***leading coefficient*** is the coefficient of the term with highest degree.

Any polynomial can be written in the form below:

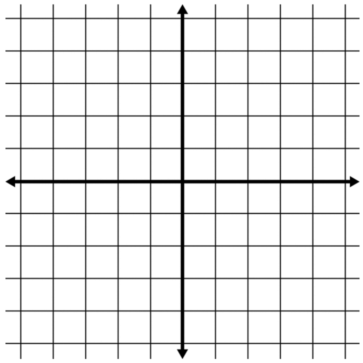
Leading coefficient *Highest degree term first to lowest degree term last* *Constant Term*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x^1 + a_0$$

Degree of Polynomials		examples	Number of Terms	examples
Linear	degree 1			
Quadratic	degree 2		***Monomial***	
Cubic	degree 3		***Binomial***	
Quartic	degree 4		***Trinomial***	
Quintic	degree 5			

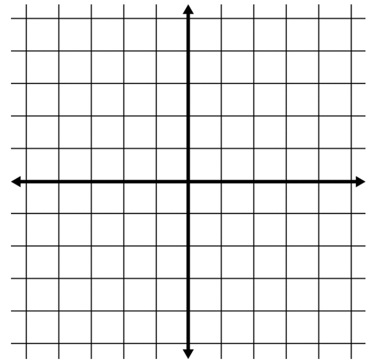
ODD degree

Leading coefficient positive



- End behavior
- *y*-intercept
- Number of *x-intercepts*
- Domain
- Range
- Maximum/minimum

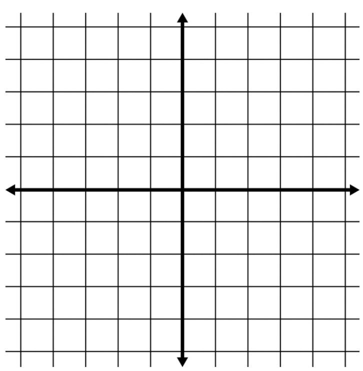
Leading coefficient negative



- End behavior
- *y*-intercept
- Number of *x-intercepts*
- Domain
- Range
- Maximum/minimum

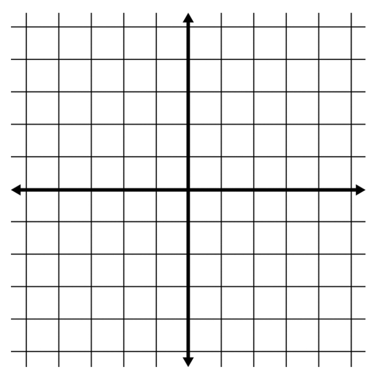
EVEN degree

Leading coefficient positive



- End behavior
- *y*-intercept
- Number of *x-intercepts*
- Domain
- Range
- Maximum/minimum

Leading coefficient negative



- End behavior
- *y*-intercept
- Number of *x-intercepts*
- Domain
- Range
- Maximum/minimum

3.2 The Remainder Theorem

Long division

$$6\overline{)253}$$

Ways to write the result:	**Check:**
$253 \div 6 = 42$, remainder 1 $\frac{253}{6} = 42$, remainder 1 $\frac{253}{6} = 42 + \left(\frac{1}{6}\right)$	(Divisor)(Quotient) + Remainder = Dividend (6)(42) + 1 = 253

Using long division to divide a polynomial by a binomial

Divide $x^3 - 12x^2 - 42$ by $x - 3$.

Express the result in the form $\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$

What restrictions are there on the variable?

Verify (check) your answer.

Division Statement

The result of dividing a polynomial $P(x)$ by a binomial of the form $x - a$ is:

$$\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$$, where $Q(x)$ is the quotient and R is the remainder.

Check: $P(x) = (x-a)Q(x) + R$

original polynomial = (divisor)(quotient) + remainder

1a) Divide the polynomial $5x^3 + 3x^2 - 12$ by $x + 2$ using ***long division.***
Express the result in the form
$$\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$$

b) What restrictions are there on the variable?

c) Write the statement that can be used to check the division.

d) Verify your answer.

2. Divide the polynomial $5x^3 + 3x^2 - 12$ by $x + 2$ using ***synthetic division.***
Express the result in the form
$$\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$$

Divide, using synthetic division:
$(2x^3 + 3x^2 - 5x + 2) \div (x + 3)$

Find the value of $P(-3)$, for $P(x) = 2x^3 + 3x^2 - 5x + 2$.

Remainder Theorem:
When a polynomial, *P(x)*, is divided by a binomial, $x - a$, the remainder is $P(a)$.

If $P(a) = 0$, then the binomial $x - a$ ***is*** a factor of *P(x)*.
If $P(a) \neq 0$, then the binomial $x - a$ is ***not*** a factor of *P(x)*.

Example
a) Use the Remainder Theorem to find the remainder when $P(x) = 8x^3 + 4x^2 - 19$ is divided by $x + 2$

b) Check your answer by using synthetic division.

c) Use the Remainder Theorem to find the remainder when $P(x) = 8x^3 + 4x^2 - 19$ is divided by $x - 1$.

3.3 The Factor Theorem

For each binomial below, find the remainder when $P(x) = x^3 - 4x^2 + x + 6$ is divided by the binomial. Which of the following binomials are factors of $P(x) = x^3 - 4x^2 + x + 6$?

a) $x+1$

d) $x-1$

b) $x+2$

e) $x-2$

c) $x+3$

f) $x-3$

Factor Theorem: $x - a$ is a factor of a polynomial, *P(x)*, if and only if $P(a) = 0$

Example

Consider the polynomial: $P(x) = x^3 + 7x^2 - 28x + 20$.

a) Given that $P(-10) = 0$, what binomial must be a factor of $P(x)$?

b) Factor $P(x)$ completely.

Integral Zero Theorem:

If $x - a$ is a factor of a polynomial with integral coefficients, $P(x)$, then a must divide evenly into the ***constant term*** of the polynomial $P(x)$.

Example

Factor fully without using technology: $2x^3 - 5x^2 - 4x + 3$

a) According to the integral zero theorem, which values could possibly give factors of this polynomial?

b) Use the remainder and factor theorems to find a factor.

c) Use either long division or synthetic division to divide $2x^3 - 5x^2 - 4x + 3$ by the factor found in part (b).

d) What is the fully factored form of $2x^3 - 5x^2 - 4x + 3$?

e) Consider the equation $2x^3 - 5x^2 - 4x + 3 = 0$, what are the solutions to this equation?

f) Graph $2x^3 - 5x^2 - 4x + 3$ on a graphing calculator. Find the values of its x-intercepts.

3.4 Equations and Graphs of Polynomial Functions

As we just saw, the solutions of an equation match up with the x-intercepts of the graph.

Example

a) Graph the function $f(x) = x^4 + x^3 - 10x^2 - 4x + 24$ using graphing technology. What are its x-intercepts?

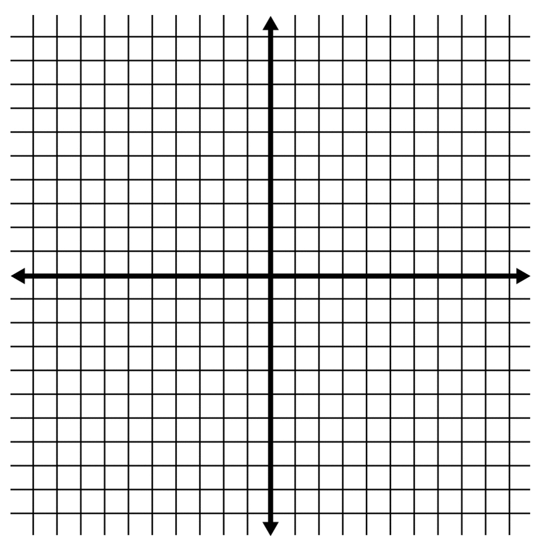
b) Use the results from part (a) to help fully factor $f(x) = x^4 + x^3 - 10x^2 - 4x + 24$.

c) What are the solutions to the equation: $x^4 + x^3 - 10x^2 - 4x + 24 = 0$

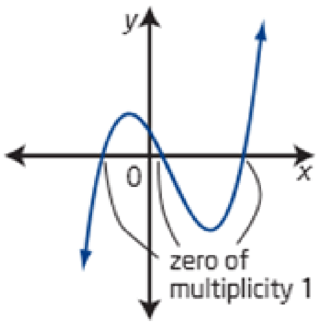
Example

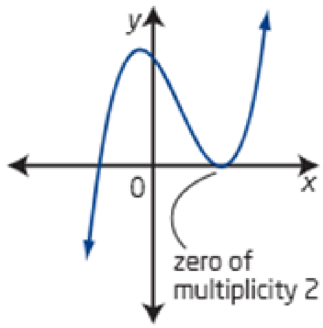
Factor completely, then analyze and sketch the graph of this polynomial function ***without*** using technology: $f(x) = x^3 + 3x^2 - 6x - 8$

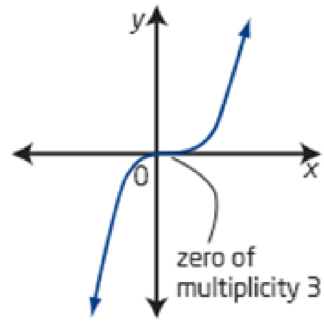
Degree	
Leading coefficient	
End behavior	
Zeros/*x*-intercepts	
y-intercept	
Interval(s) where function is positive or negative	



Multiplicity of a zero – the multiplicity of a zero is the number of times a zero of a polynomial occurs.







Example
Find the equation for the given graph.