Chapter 4: Trigonometry and the Unit Circle

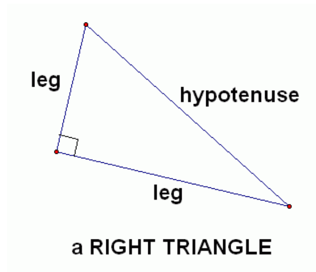
4.0 Trigonometry Review

Trigonometry is the study of triangles and trigonometric functions. First, we review some trigonometry dealing with triangles.

Triangles

- have three angles, the measures add up to 180°
- longest side of triangle is across from the largest angle
- shortest side of a triangle is across from the smallest angle

- ***right triangles*** are triangles that have a ***right angle*** (90°)
  - ***hypotenuse*** is the longest side of a right triangle
  - other sides of the triangle are often called ***legs***
  - hypotenuse is always **across** from the right angle
  - in a right triangle, we can use the ***Pythagorean Theorem***



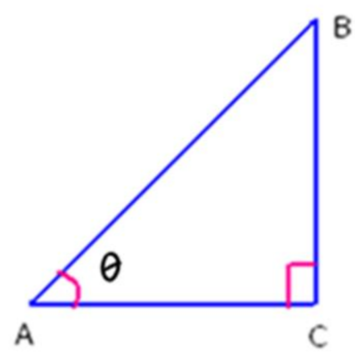
If the two legs of a right triangle are called a and b, and the hypotenuse is called c, then $a^2 + b^2 = c^2$.

Look at the right triangle shown below. The angle by point A is labeled with the Greek letter θ, read "theta." Angles are very commonly labeled with the letter θ.

Which side is the hypotenuse?

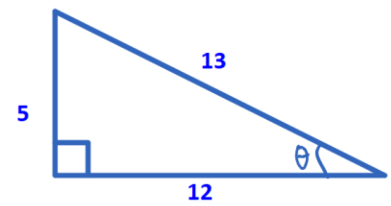
Which side is opposite θ?

Which side is adjacent to θ?



When we know the lengths of the sides of a right triangle, we can calculate **ratios** that compare the lengths of two different sides.

First, we label the hypotenuse, and the sides that are opposite and adjacent to angle θ .



There are six different ratios one can create. We'll leave the ratios in fractional form.

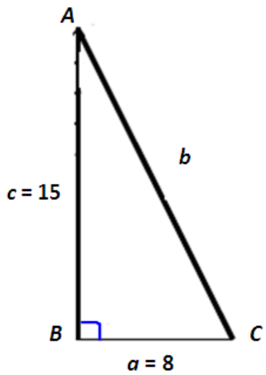
$$\frac{opposite}{hypotenuse} = \qquad \frac{adjacent}{hypotenuse} = \qquad \frac{opposite}{adjacent} =$$

$$\frac{hypotenuse}{opposite} = \qquad \frac{hypotenuse}{adjacent} = \qquad \frac{adjacent}{opposite} =$$

These ratios are called the ***trigonometric ratios***. Knowing them makes it possible to find the measure of each angle in the triangle.

<u>***Primary Trigonometric Ratios***</u>	<u>***Reciprocal Trigonometric Ratios***</u>
$SINE = \sin\theta = \frac{opposite}{hypotenuse} = \frac{O}{H}$	$COSECANT = \csc\theta = \frac{hypotenuse}{opposite} = \frac{H}{O}$
$COSINE = \cos\theta = \frac{adjacent}{hypotenuse} = \frac{A}{H}$	$SECANT = \sec\theta = \frac{hypotenuse}{adjacent} = \frac{H}{A}$
$TANGENT = \tan\theta = \frac{opposite}{adjacent} = \frac{O}{A}$	$COTANGENT = \cot\theta = \frac{adjacent}{opposite} = \frac{A}{O}$

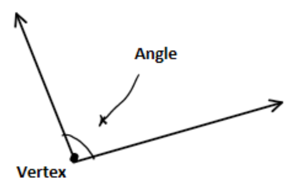
Example Find the measure of each side and angle (correct to nearest degree) in the right triangle shown below.



4.1 <u>Angles and Angle Measure</u>

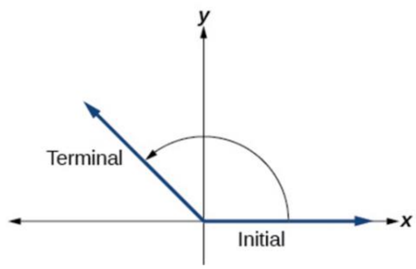
Trigonometry does a lot more than solve triangles. It can be used to analyze many ***repeating patterns*** – things like sound, light, ocean tides, and circular motion.

We start by looking carefully at **ANGLES**.
Remember, angles measure the space between two rays that meet at the ***vertex*** of the angle.



Angles in ***standard position***

- Have the vertex at the origin (0, 0)
- Have a specific direction of rotation, shown with an arrow.
- Have the ***initial arm*** on the positive *x*-axis
- Have the ***terminal arm*** either in one of the four quadrants, or on the *x*- or *y*-axis.

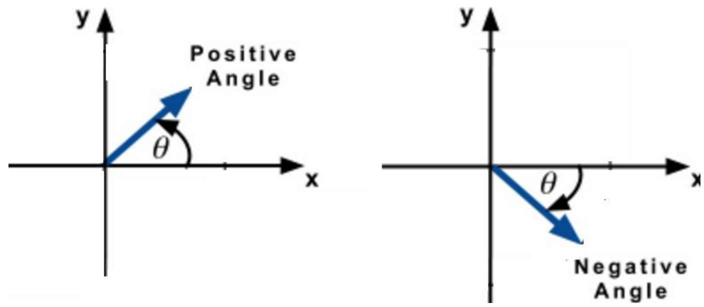


Wherever the terminal arm is, that's how we decide what to call an angle. Options are:

- First quadrant angle
- Second quadrant angle
- Third quadrant angle
- Fourth quadrant angle
- Quadrantal angle

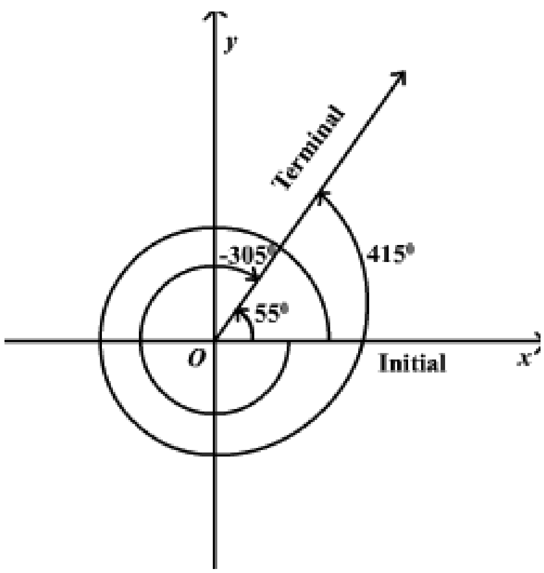
positive angles start on the positive *x*-axis, and rotate counter-clockwise

negative angles start on the positive *x*-axis, and rotate clockwise



Coterminal Angles are

- different in size but
- terminate in the same place



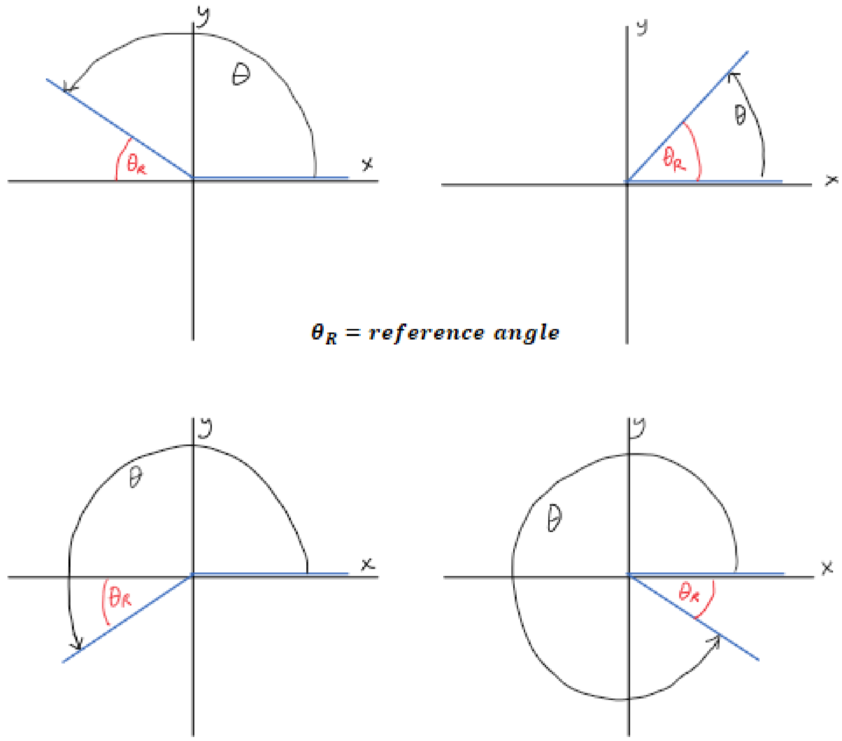
Find another positive angle and another negative angle that are coterminal to the shown angles.

General form for coterminal angles – this is an expression that generates ALL the angles that are coterminal to a specific angle. Here's how we write the angles coterminal to 55° in general form:

reference angle θ_R of a standard position angle θ

- is the smallest angle formed between the terminal arm of θ and the x-axis

- is positive



Try this

1. Draw each angle in standard position. (Estimate – you don't need to use a protractor.)

a) 110° b) -40° c) 270°

2. Find a positive and a negative coterminal angle to the angle 160°.

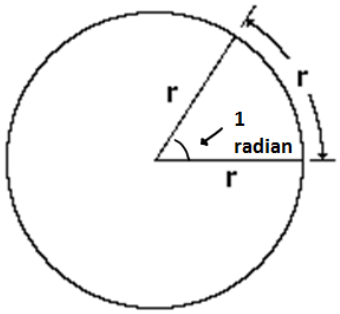
3. Give the general expression for ALL angles coterminal to the angle 25°.

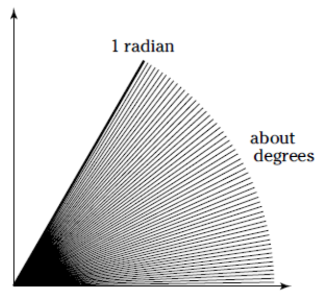
4. Find the reference angle for each angle.

a) 235° b) 310° c) 110°

Another unit used to measure angles (besides degrees) is ***radians***. We need to know how to work with radians, as they make some calculus questions much easier (and this is PreCalculus, after all!)

1 *radian* = the measure of an angle that cuts off a length on the circle equal in length to the circle's radius.





Try this

Sketch a standard position angle measuring:

a) 1 radian

b) 2 radians

c) 3 radians

b) 6 radians

How many radians are in a full rotation? (Think about how many radius lengths will fit onto the full circumference of a circle.)

$\_\_\_\_\_\ radians = a\ complete\ rotation\ = \_\_\_\_\_^{\circ}$

$\_\_\_\_\_\ radians = a\ straight\ angle\ = \_\_\_\_\_^{\circ}$

Common Angles
Some angles are used so frequently that it is very helpful to simply KNOW their measurement in both radians and degrees.

$\pi =$ $\qquad 2\pi =$ $\qquad \dfrac{\pi}{2} =$ $\qquad \dfrac{\pi}{3} =$

$\dfrac{\pi}{4} =$ $\qquad \dfrac{\pi}{6} =$ $\qquad 0 =$ $\qquad \dfrac{3\pi}{2} =$

Converting Units
We know that 30 minutes is the same thing as ½ an hour. But what about 7452 minutes? What is that, in hours? Here's one way to change units – multiply by a factor of "1"

Converting Angle Measure

For angles that are not the common ones listed above, we convert angle measurements between degrees and radians by multiplying by the appropriate conversion unit.

$$(\text{degrees}) \times \left(\frac{\pi}{180^\circ}\right) = \text{radians}$$

$$(\text{radians}) \times \left(\frac{180^\circ}{\pi}\right) = \text{degrees}$$

Try these
Convert from degrees to radians. Express answer correct to 2 decimal places.

425°

Convert from degrees to radians. ***Leave answer as a simplified fraction, in terms of π.***

-330°

Convert from radians to degrees.

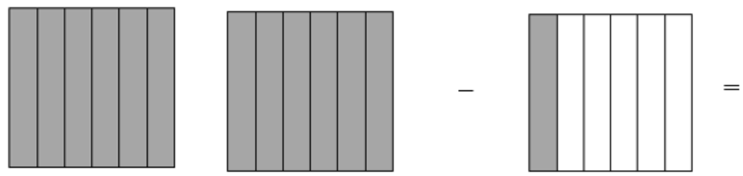
a) $\dfrac{3\pi}{8}$

b) 2 radians

Working with Radians in Fraction Form

Because π radians is the size of a straight angle (half a rotation), we end up working a lot with angles written as fractional parts of π. Let's review adding/subtracting fractions.

$2-\frac{1}{6}=$



Try

a) $1+\frac{1}{4}=$

b) $2-\frac{1}{3}=$

c) $1-\frac{1}{6}=$

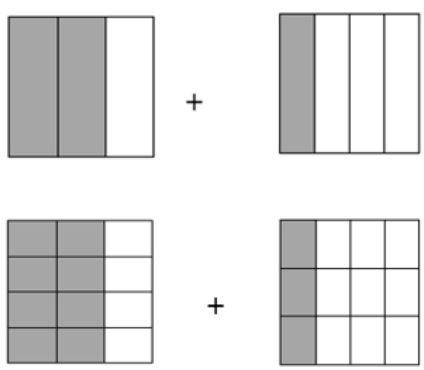
d) $2-\frac{1}{4}=$

e) $1-\frac{1}{4}=$

f) $1+\frac{1}{3}=$

Different Denominators

$\frac{2}{3}+\frac{1}{4}=$



Try

a) $\frac{1}{2}+\frac{1}{3}=$

b) $\frac{7}{6}+\frac{1}{4}=$

c) $\frac{3}{5}+\frac{1}{7}=$

Is π in the fraction? It still works the same way!

Try

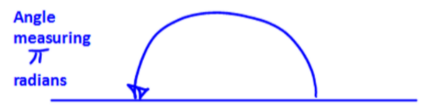
a) $\pi+\frac{\pi}{6}=$

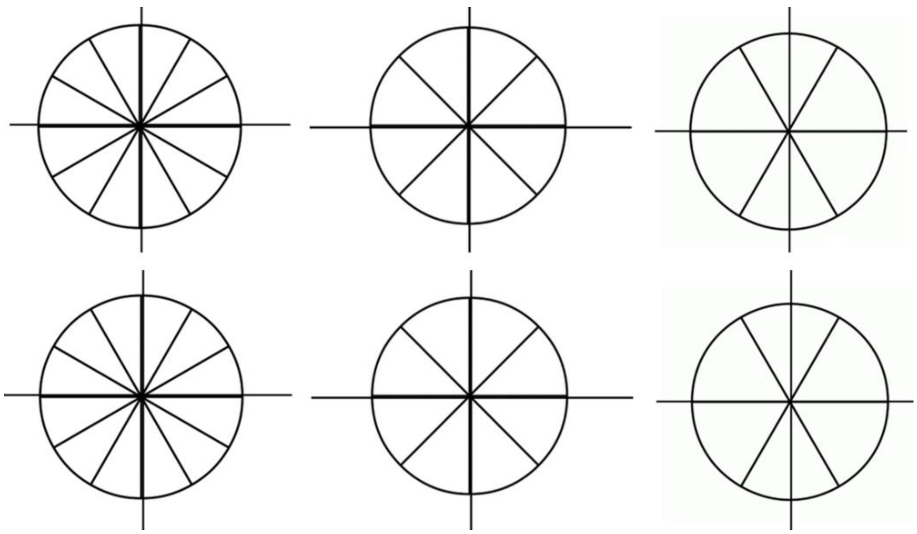
b) $2\pi-\frac{\pi}{2}=$

c) $\frac{3\pi}{4}+2\pi=$

Standard-Position Angles in Radian Measure

A straight angle measures π radians. This helps us when sketching angles that are fractions involving π.





For each angle below:

- Draw the angle in standard position.
- List two angles that are coterminal to the given angle. Use radians, not degrees!

a) $\frac{5\pi}{6}$

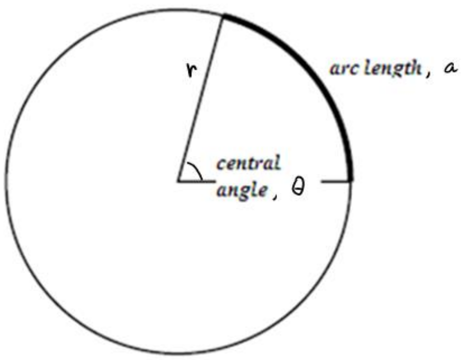
b) $-\frac{\pi}{4}$

c) $\frac{5\pi}{3}$

d) $\frac{7\pi}{6}$

Arc Length

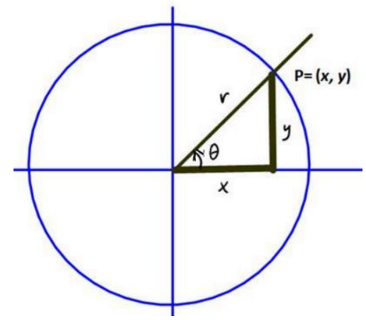
Suppose we have a circle with radius = 20 cm. If we mark off a central angle measuring 80°, what arc length (length along the circle's circumference) does the angle cut off?



Suppose we have another circle, this one with diameter 9 cm. If we know that an arc measuring 7 cm is subtended by a central angle, what is the measure of that central angle, in radians? What is the measure of the central angle, in degrees?

4.2 The Unit Circle

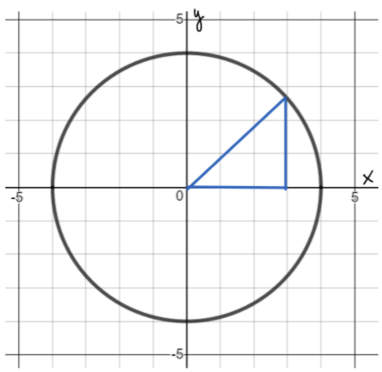
A circle is the set of all points that are a certain distance, ***radius***, from a given point, the center. Using the Pythagorean Theorem, we can get an equation for a circle.



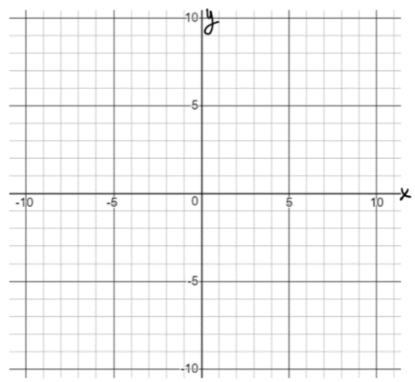
The equation for a circle with center $(0,0)$ and radius ***r*** is:

Try

a) Find the equation of this circle.

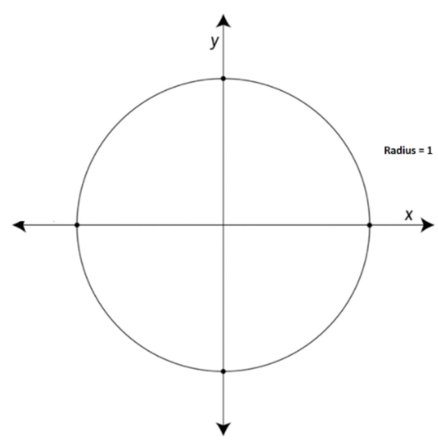


b) Sketch the graph of $x^2 + y^2 = 64$



Unit Circle

If we choose r = 1, we get a circle with radius 1 unit in length. This is called the ***unit circle***, and its equation is $x^2 + y^2 = 1$.

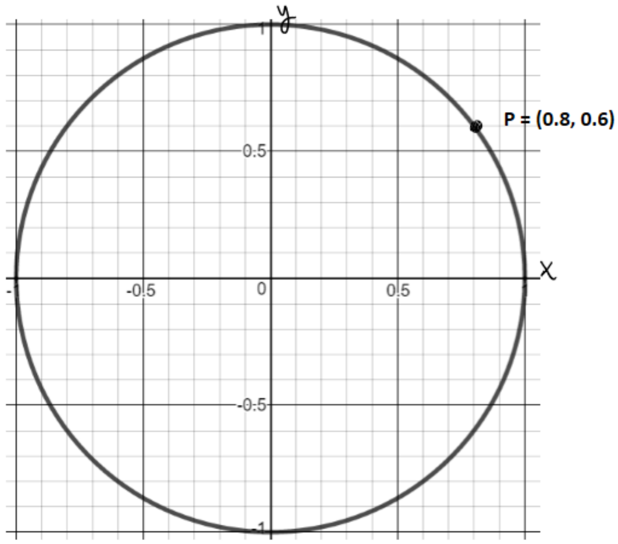


a) Is the point $(0.6,\ 0.4)$ on the unit circle?

b) The point below is on the unit circle. Use the unit circle equation, $x^2 + y^2 = 1$, to find the value of the unknown coordinate.

$$\left(x,\ \frac{1}{2}\right)$$

On the unit circle below, we have a point *P*, with coordinates (0.8, 0.6). We draw a line segment connecting *P* to the origin, (0, 0). This radius and the *x*-axis form a standard position angle, which we call θ. Because this is an accurate drawing, we could use a protractor and get the size of angle θ – it is about 36.87°. By drawing in a line segment that connects *P* to the *x*-axis, we create a right-triangle, with the right-angle on the *x*-axis.



From the diagram, we get:

$$\cos\theta = \frac{adj}{hyp} =$$

$$\sin\theta = \frac{opp}{hyp} =$$

Using the calculator, we get:

$\cos 36.87^\circ =$

$\sin 36.87^\circ =$

Let $P(\theta) = (x, y)$ be the point where the terminal arm of a standard-position angle θ intersects the **unit circle**. Then we know:

- the *x*-coordinate's value is equal to the cosine of the angle $\qquad x = \cos\theta$
- the *y*-coordinate's value is equal to the sine of the angle $\qquad y = \sin\theta$

We now have a way to find sine and cosine values for ANY angle, including:

- negative angles
- 0°
- angles larger than 90°

Using the triangle definitions (SOHCAHTOA) for those types of angles doesn't really make sense. For example, what would be the adjacent, opposite, and hypotenuse lengths for an angle measuring 0° ?

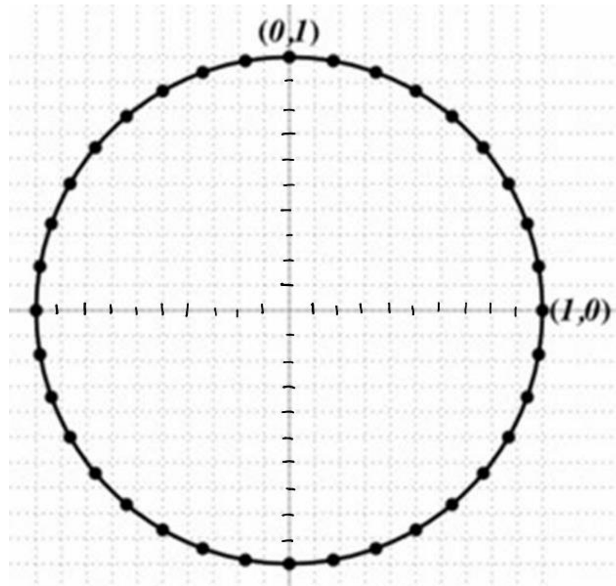
Try – NO calculator. *(You don't need it! You can figure them out yourself!)*

$P(0^\circ) =$ $\qquad \cos(0^\circ) =$ $\qquad \sin(0^\circ) =$

$P\left(\frac{3\pi}{2}\right) =$ $\qquad \sin\left(\frac{3\pi}{2}\right) =$ $\qquad \cos\left(\frac{3\pi}{2}\right) =$

Finding Approximate Values of Trigonometric Ratios

Estimate each value using the graph at right. Compare with the calculator answer, correct to 4 decimal places.



a) $\cos 250^\circ =$ $\qquad$ $\sin 250^\circ =$

b) $\cos 500^\circ =$ $\qquad$ $\sin 500^\circ =$

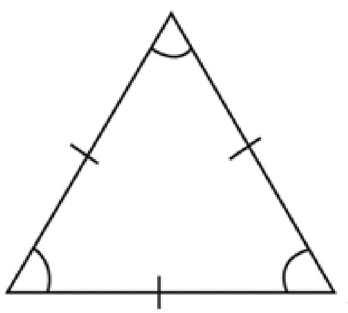
c) $\cos\left(-10^\circ\right) =$ $\qquad$ $\sin\left(-10^\circ\right) =$

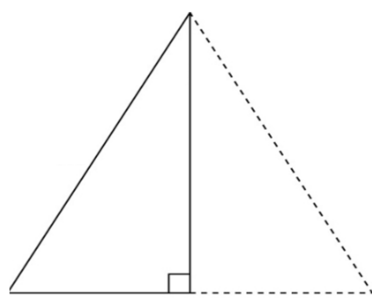
http://www.malinc.se/math/trigonometry/unitcircleen.php

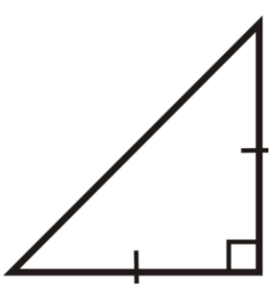
Special Triangle Angles

Besides the quadrantal angles, there are some other angles for which we can find ***exact*** coordinates for $P(\theta)$. These angles relate to special triangles.

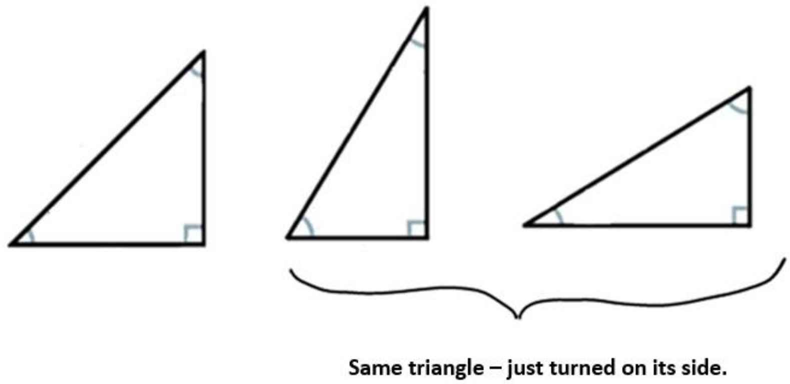
Remember special triangles?

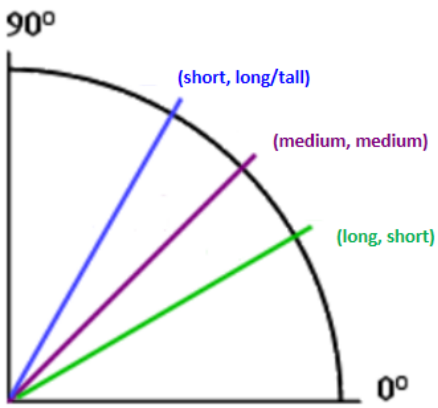






Let's use those triangle angles in the unit circle setting. We need to adjust the size of the triangles, making their hypotenuse length = 1.





Short side length =

Medium side length =

Tall/long side length =

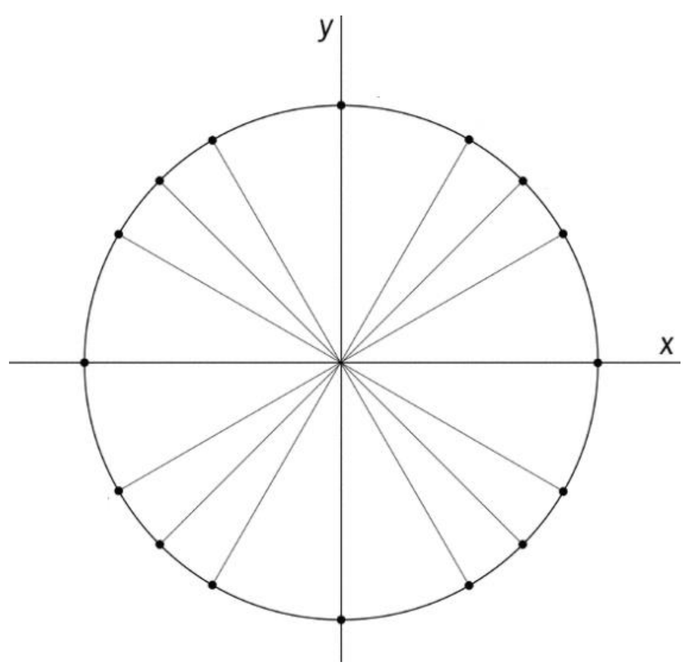
We know that the shortest side of a triangle is across from its smallest angle. This helps us label the coordinates correctly for different angles.

Try – NO calculator. Get the ***exact values***.

$\cos\left(30^\circ\right)=$ $\sin\left(30^\circ\right)=$

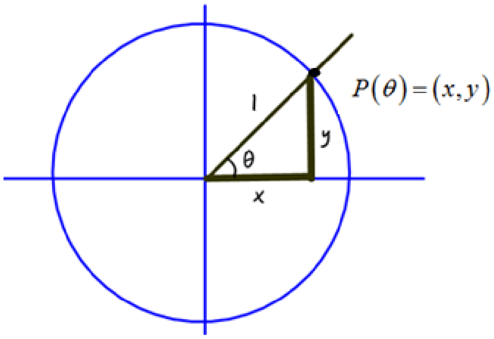
$\sin\left(135^\circ\right)=$ $\cos\left(135^\circ\right)=$

$\cos\left(300^\circ\right)=$ $\sin\left(300^\circ\right)=$



4.3 Trigonometric Ratios

We can use the unit circle diagram to get definitions for all six of the trigonometric ratios.



Primary Ratios ***Reciprocal Ratios***

$\sin\theta =$

$\cos\theta =$

$\tan\theta =$

More Exact Values **-** Use the unit circle to find each ***exact value – no calculator!***

a) $\cos(\pi)$ $\sin(\pi)$ $\tan(\pi)$

$\sec(\pi)$ $\csc(\pi)$ $\cot(\pi)$

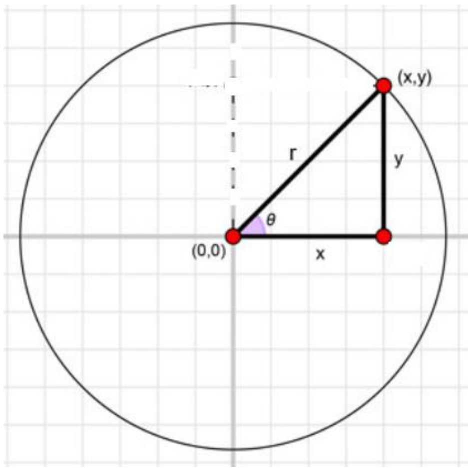
b) $\cos\left(\frac{7\pi}{4}\right)$ $\sin\left(\frac{7\pi}{4}\right)$ $\tan\left(\frac{7\pi}{4}\right)$

$\sec\left(\frac{7\pi}{4}\right)$ $\csc\left(\frac{7\pi}{4}\right)$ $\cot\left(\frac{7\pi}{4}\right)$

c) $\cos\left(\frac{7\pi}{6}\right)$ $\sin\left(\frac{7\pi}{6}\right)$ $\tan\left(\frac{7\pi}{6}\right)$

$\sec\left(\frac{7\pi}{6}\right)$ $\csc\left(\frac{7\pi}{6}\right)$ $\cot\left(\frac{7\pi}{6}\right)$

Sometimes the radius is not 1. Here are the ratio definitions that work for any *r* value.



Primary Ratios	***Reciprocal Ratios***
$\sin\theta = \frac{y}{r}$	
$\cos\theta = \frac{x}{r}$	
$\tan\theta = \frac{y}{x}$	

Know these definitions!

Example
The terminal arm of a standard position angle θ contains the point $(-2,-5)$. Find the value of all six trigonometric ratios for angle θ. You do not need to find the size of angle θ. Leave answers in ***exact fractional form***.

The angle terminates in quadrant _________

x = ______

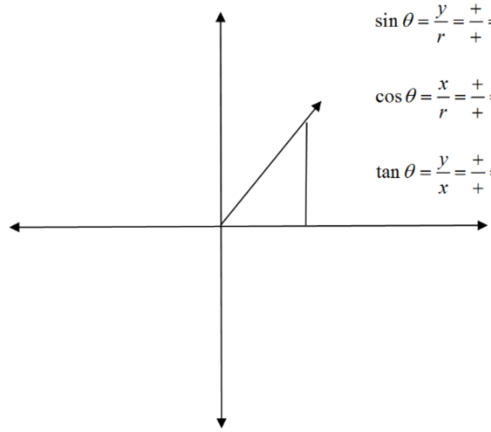
y = ______

r = ______

Finding the Signs of the Trigonometric Ratios

How can we predict whether a specific trigonometric ratio will be positive or negative?

r - positive in every quadrant
x - depends on the quadrant, can be + or –
y - depends on the quadrant, can be + or –



$\sin\theta = \frac{y}{r} = \frac{+}{+} = positive$

$\cos\theta = \frac{x}{r} = \frac{+}{+} = positive$

$\tan\theta = \frac{y}{x} = \frac{+}{+} = positive$

Example

Given the information below, decide in which quadrant (or quadrants) angle θ can terminate.

a) $\tan\theta < 0$

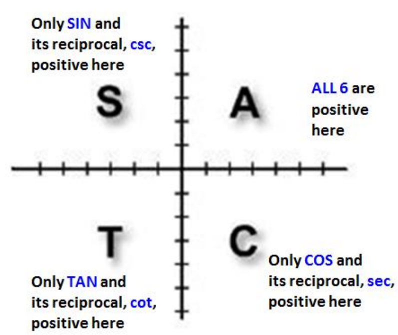
b) $\csc > 0$

c) $\sin\theta < 0$ *and* $\cot\theta > 0$

d) $\sec\theta > 0$ *and* $\tan\theta < 0$

Example

Find the value of $\sec\theta$, if we know $\tan\theta = -\frac{4}{3}$ and $\sin\theta > 0$.



Approximate Values

For angles not related to special angles, calculators can give us accurate approximations.

Try

Evaluate each of the following ratios correct to 4 decimal places, using a calculator.

a) $\tan\left(-65^\circ\right) =$

b) $\sec 417^\circ =$

c) $\cot\left(\frac{3\pi}{5}\right) =$

d) $\cos\left(4\right) =$

e) $\sin\left(-200^\circ\right) =$

f) $\csc\left(\frac{4\pi}{7}\right) =$

Use the correct mode – either degrees or radians, matching the angle's units.

Be careful when evaluating reciprocal ratios.
Don't take the reciprocal of the angle!
Instead, use these relationships:

$$\csc\left(angle\right) = \frac{1}{\sin(angle)}$$

$$\cot\left(angle\right) = \frac{1}{\tan(angle)}$$

$$\sec\left(angle\right) = \frac{1}{\cos(angle)}$$

4.4 Solving Trigonometric Equations

We now know how to find trigonometric ratio values for any angle. Sometimes we must settle for an approximation, other times we can give an exact value.

For example:

$\sin 47^\circ =$ $\cos\left(\frac{5\pi}{6}\right) =$

Now we look at the opposite situation. Given the value of a trigonometric ratio, how do we find out the size of the ***angle***? This is a process that works well:

Isolate – Decide – Get Reference Angle - Solve

Examples Solve algebraically. If possible, give exact answers. Otherwise, give answers correct to one decimal place.

a) $\sin\theta - 0.8 = 0$, for $0^\circ \le \theta < 360^\circ$

b) $\cos\theta = -1$, for $-\pi \le \theta < 2\pi$

c) $3\cos\theta - 12 = 0$, for $0 \le \theta < 2\pi$

d) $3\tan\theta - 3 = 0$, for $0 \le \theta < 2\pi$

e) $3\sec\theta - 10 = 0$, for $0^\circ \le \theta < 720^\circ$

Isolate – Decide – Get Reference Angle - Solve

1) Isolate the trigonometric term. If it uses cot, sec, or csc, take the ***reciprocal*** of both sides of the equation to get a simpler-to-solve version of the equation.

2) Decide whether the equation can be solved using

- special angles on the unit circle. Look for: $0, \pm 1, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{3}}{2}$
- the $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ button on the calculator
- OR, cannot be solved

3) Determine in which quadrants answers will be found.

4) Find the reference angle and use it to find all the solutions in the given domain. Use the same units (either degrees or radians) as shown in the question's domain.

Examples Solve algebraically. If possible, give exact answers. Otherwise, give answers correct to one decimal place.

a) $\sin\theta = -0.8$, for $0^\circ \le \theta < 360^\circ$

If we calculate $\sin^{-1}(-0.8)$, the calculator gives us a negative answer. We don't want this, because the domain asks for only positive answers.
To avoid this problem, give the calculator the trigonometric ratio as a positive quantity. This guarantees the calculator will give us the ***reference angle***, which will be positive.

b) $5\cos\theta - 2 = 2\cos\theta - 4$, for $0^\circ \le \theta < 720^\circ$

General Solution

How many solutions a trigonometric equation has depends on the domain specified in the question. When the domain is all real numbers, there are infinitely many solutions. This is called the ***general solution***.

Example Solve algebraically. If possible, give exact answers. Otherwise, give answers correct to one decimal place.

a) $3\cos\theta - 1 = 0$, ***general solution*** in degree measure.

b) $\cot\theta + 5 = 0$, ***general solution*** in radian measure.

Here's how to write the ***general solution***:

- list each answer θ, found in one full rotation, separately
- to each answer add on the appropriate amount, either

$$+2\pi n,\ n \in I \qquad \text{or} \qquad +360^{\circ} n,\ n \in I$$

For equations using *tangent* or *cotangent*, we find that in one full rotation the two solutions are spaced exactly π or 180° apart. Because of this, we can just write the first solution, and add onto it:

$$+\pi n,\ n \in I \qquad \text{or} \qquad +180^{\circ} n,\ n \in I$$

Example

Suppose that for a certain equation, we are all told its solutions for $0^{\circ} \le \theta < 360^{\circ}$ are $\theta = 20^{\circ}$ and $\theta = 160^{\circ}$. What is the ***general solution***?

Solving Second-Degree Trigonometric Equations

When we solve equations with an exponent we usually start by factoring.

For example, solve: $2x^2 - 1 = x$

Example

Solve algebraically. If possible, give exact answers. Otherwise, give answers correct to one decimal place.

$$2\tan^2\theta - 1 = \tan\theta \text{, for } 0^\circ \le \theta < 360^\circ$$

Example

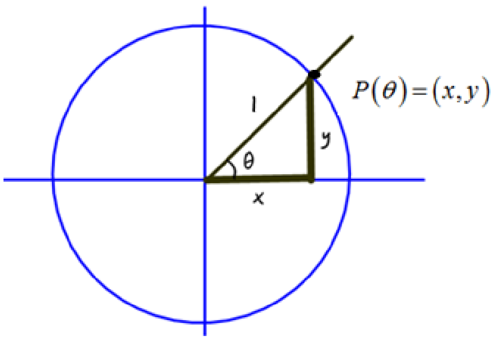
Solve algebraically. If possible, give exact answers. Otherwise, give answers correct to one decimal place.

$2\cos^2\theta - 1 = 0$, for $0 \le \theta < 2\pi$

Chapter 5: Trigonometric Functions and Graphs

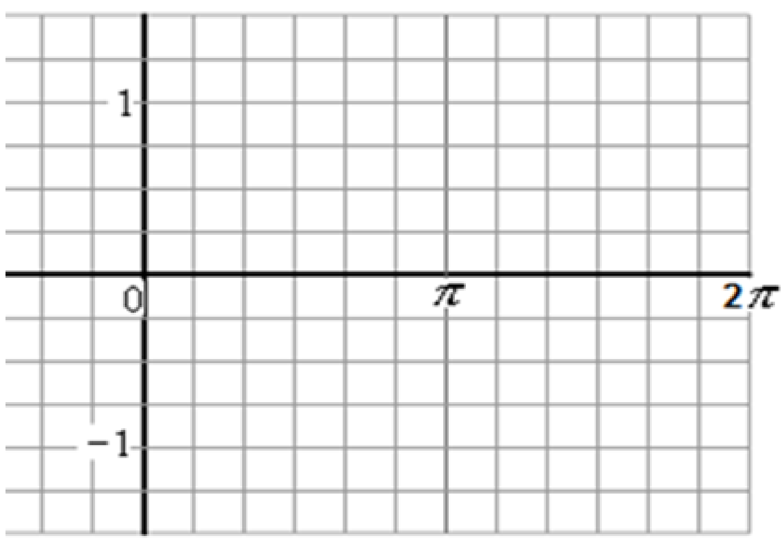
5.1 Graphing Sine and Cosine Functions

Let's track what happens to $P(\theta)$ as θ, a standard-position angle, gets larger.



		$y = \cos\theta$
Q1	As θ increases from 0 to $\frac{\pi}{2}$	cosine values (*x*-values)
Q2	As θ increases from $\frac{\pi}{2}$ to π	cosine values (*x*-values)
Q3	As θ increases from π to $\frac{3\pi}{2}$	cosine values (*x*-values)
Q4	As θ increases from $\frac{3\pi}{2}$ to 2π	cosine values (*x*-values)

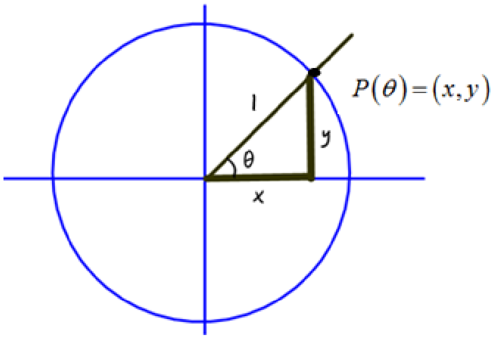
$y = \cos\,\theta$ (also often written as $y = \cos\,x$)



θ	$\cos\,\theta$
0	
$\frac{\pi}{6}$	
$\frac{2\pi}{6} = \frac{\pi}{3}$	
$\frac{3\pi}{6} = \frac{\pi}{2}$	
$\frac{4\pi}{6} = \frac{2\pi}{3}$	
$\frac{5\pi}{6}$	
$\frac{6\pi}{6} = \pi$	
$\frac{9\pi}{6} = \frac{3\pi}{2}$	
$\frac{12\pi}{6} = 2\pi$	

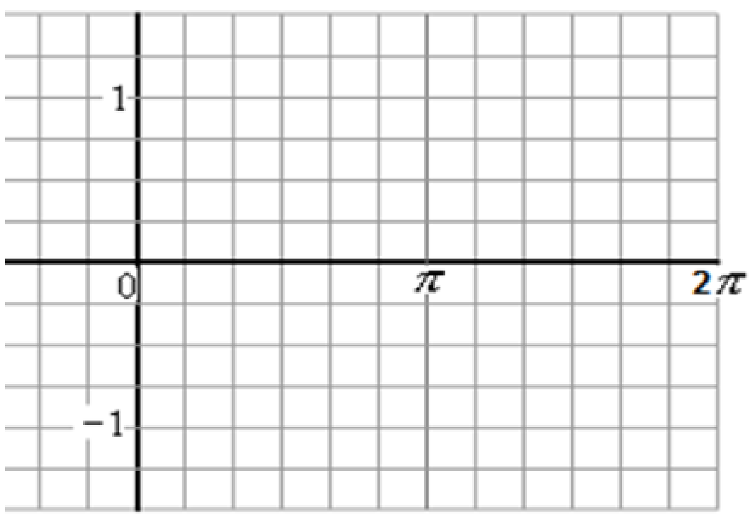
Maximum: Minimum: Range: Amplitude:

Domain: *x*-intercepts Period: Center line equation:



		$y = \sin\theta$
Q1	As θ increases from 0 to $\frac{\pi}{2}$	sine values (y-values)
Q2	As θ increases from $\frac{\pi}{2}$ to π	sine values (y-values)
Q3	As θ increases from π to $\frac{3\pi}{2}$	sine values (y-values)
Q4	As θ increases from $\frac{3\pi}{2}$ to 2π	sine values (y-values)

$y = \sin\,\theta$ (also often written as $y = \sin x$)



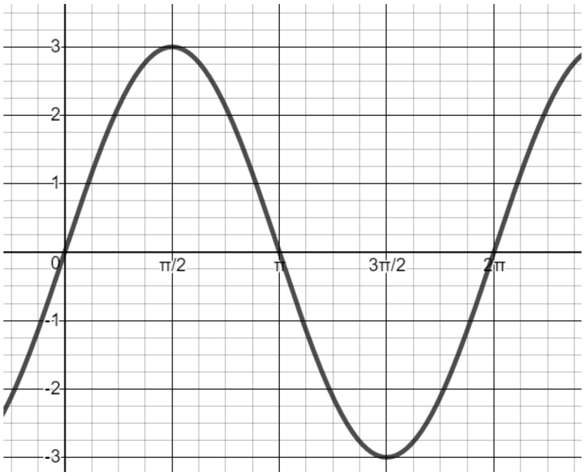
θ	$\sin\,\theta$
0	
$\frac{\pi}{6}$	
$\frac{2\pi}{6} = \frac{\pi}{3}$	
$\frac{3\pi}{6} = \frac{\pi}{2}$	
$\frac{4\pi}{6} = \frac{2\pi}{3}$	
$\frac{5\pi}{6}$	
$\frac{6\pi}{6} = \pi$	
$\frac{9\pi}{6} = \frac{3\pi}{2}$	
$\frac{12\pi}{6} = 2\pi$	

Maximum: Minimum: Range: Amplitude:

Domain: x-intercepts Period: Center line equation:

Amplitude

is the **vertical** distance from the center line of a trigonometric graph to its maximum or minimum. The untransformed graphs of $y = \sin x$ and $y = \cos x$ have amplitude 1.



For $y = a\cos x$ or $y = a\sin x$

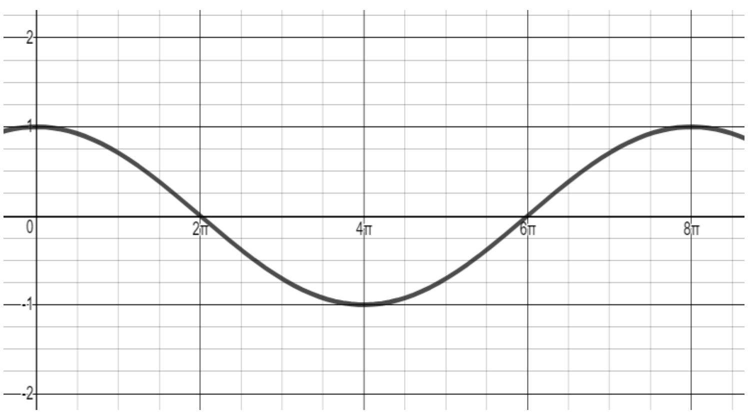
- vertical stretch, factor a
- amplitude = $|\,a\,|$
- if $a < 0$, graph is reflected across x-axis
- amplitude = $\dfrac{|\max - \min|}{2}$

Amplitude for graph shown at left?

Equation of the graph?

Period

is the **horizontal** length of one complete cycle. The untransformed graphs of $y = \sin x$ and $y = \cos x$ have a period length of 2π (or 360°, if working in degree measure).



For $y = \cos(bx)$ or $y = \sin(bx)$

- horizontal stretch, factor $\dfrac{1}{b}$
- period = $\dfrac{2\pi}{|\,b\,|}$ or $\dfrac{360^\circ}{|\,b\,|}$
- if $b < 0$, graph is reflected across y-axis

Period for graph shown at left?

Equation of the graph?

Since

graph's actual period = $\dfrac{2\pi}{|\,b\,|}$, then $|\,b\,| = \dfrac{2\pi}{\textit{graph's actual period}}$

If working in degrees, since

graph's actual period = $\dfrac{360^\circ}{|\,b\,|}$, then $|\,b\,| = \dfrac{360^\circ}{\textit{graph's actual period}}$

When sketching a period of a trigonometric function graph, we

- multiply the period length by ¼, to determine the spacing between key points
- plot key points: maximum, minimum, and center-line
- connect key points smoothly, getting a sinusoidal shape

Try

1a) $y = -3\sin(5x)$

amplitude: period: key point spacing:

b) $y = -\frac{1}{4}\sin\left(\frac{1}{3}x\right)$

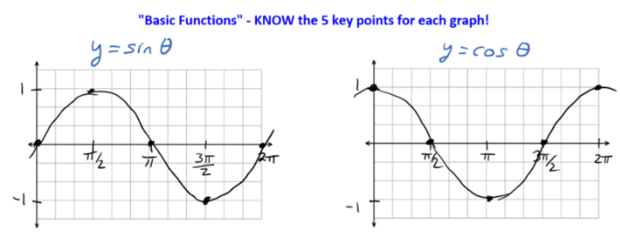
amplitude: period: key point spacing:

2. Write the equation of a function with these characteristics.

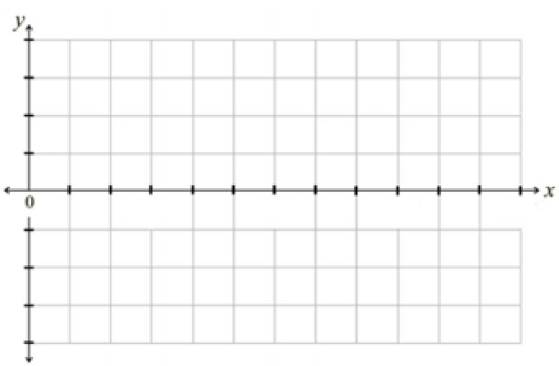
a) sine function; amp = 3, period = π

b) cosine function, amp =2.4, period = 10π

3. For each equation below, accurately sketch one period of its graph. Give the coordinates of 5 key points.

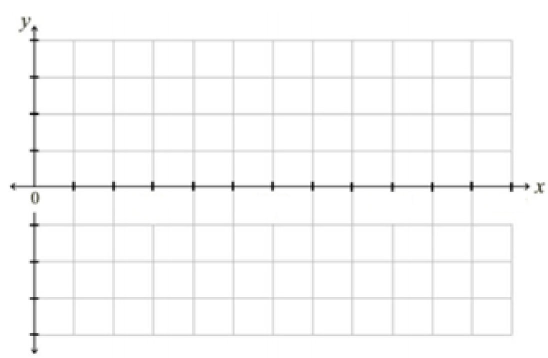


a) $y = 3\sin(2x)$



x	y

b) $y = -2\cos\left(\frac{1}{6}x\right)$

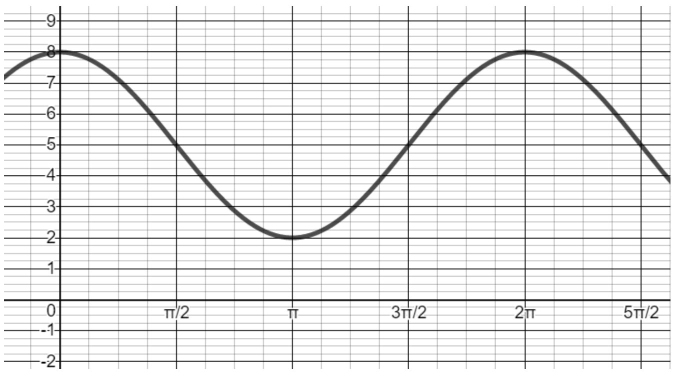


x	y

5.2 More Transformations of Sinusoidal Functions

Vertical Displacement

is the amount of vertical translation (up/down) a sinusoidal graph moves



For $y = a\cos x + d$ or $y = a\sin x + d$

- vertical displacement, d units
- center line is located at $y = d$
- when we have no equation, we can figure out the vertical displacement from the graph:

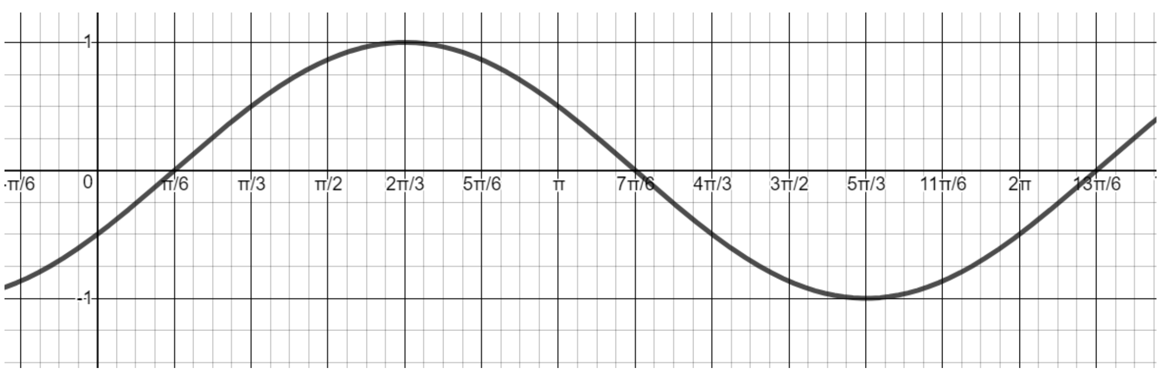
$$vertical\ \ disp = \frac{\max + \min}{2}$$

Vertical displacement for this graph?

Equation of this graph?

Phase Shift

is the amount of horizontal translation (left/right) a sinusoidal graph moves



For $y = \cos(x - c)$ or $y = \sin(x - c)$

- phase shift, c units
- when we have no equation, we use the graph to find the phase shift.
  Choose a period of either sine or cosine that begins near the y-axis. Identify how much it has moved left/right compared to the basic (untransformed) graph.

For the graph above, find its equation in the form: $y = \sin(x - c)$

For the graph above, find its equation in the form: $y = \cos(x - c)$

Summary

$$y = a\sin b(x-c)+d$$

Try

Determine the amplitude, period, phase shift and vertical displacement.

a) $y=5\cos\left(3\left(x+\dfrac{\pi}{4}\right)\right)+2$

b) $y=\dfrac{1}{2}\cos\left(x+80^\circ\right)+5$

c) $y=-2\sin\left(5x-\dfrac{\pi}{3}\right)+4$

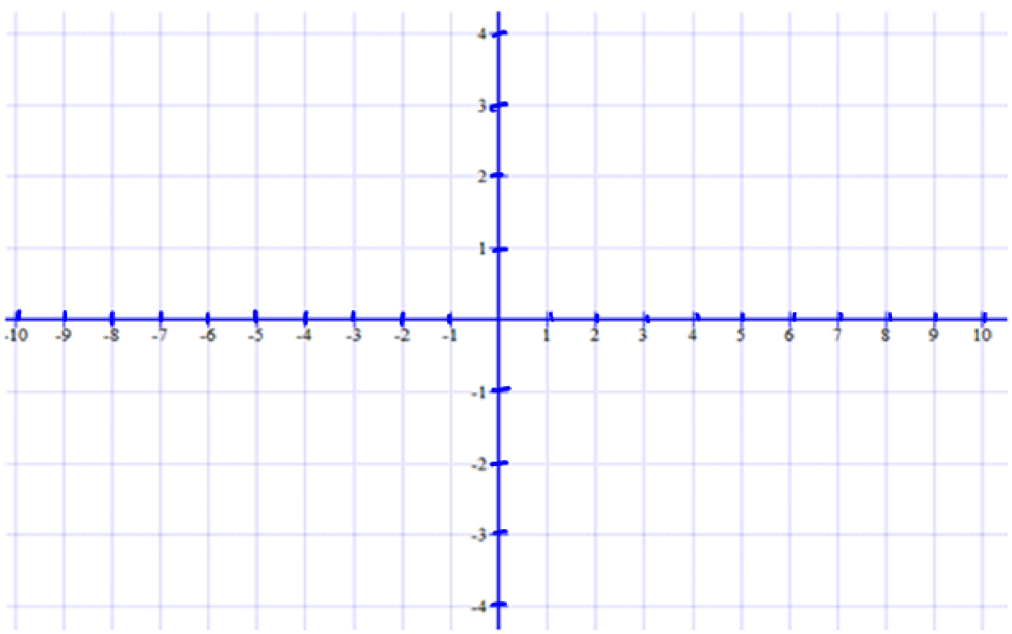
Sketching a Sinusoidal Graph

Consider the equation: $y = 3\sin\left(\frac{2\pi}{12}(x+5)\right) - 1$

a) Key features:

basic sine shape	vertical displacement	equation of center line
amplitude	maximum	minimum
period	spacing	phase shift

b) Accurately sketch one period of the graph. Give the coordinates of 5 key points. Include the center line on your sketch.



x	y

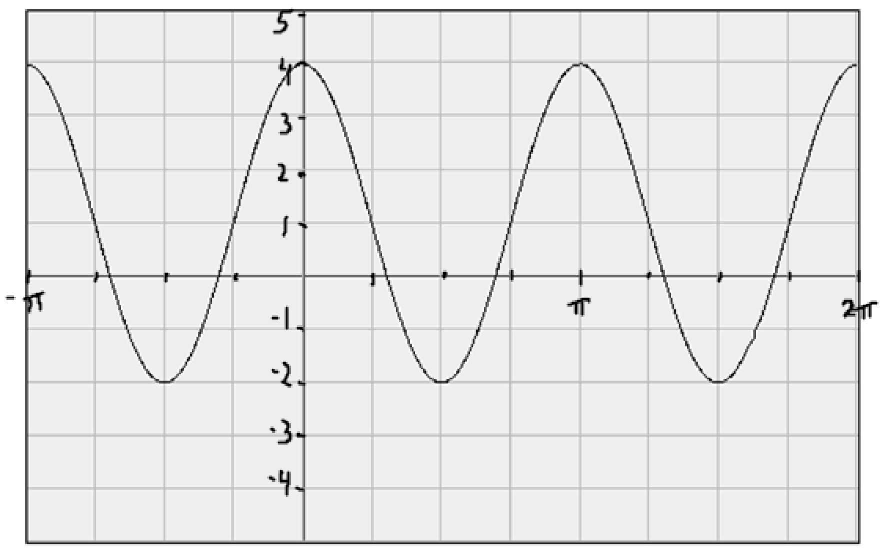
Finding the Equation of a Graph

Sine and cosine graphs are both called ***sinusoidal graphs***.

- For any sinusoidal graph, it is possible to write a sine equation that creates that graph, <u>and</u> a cosine equation that creates that same graph.
- There are many different equations that generate the same sinusoidal graph.

Example

Give two different equations that create this graph.



Maximum:

Minimum:

Center line:

Vertical displacement:

Amplitude:

Period:

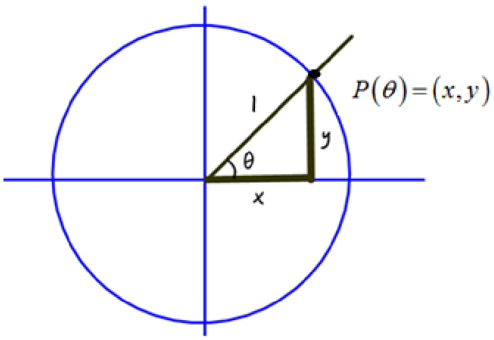
b value:

Possible sine equation:

Possible cosine equation:

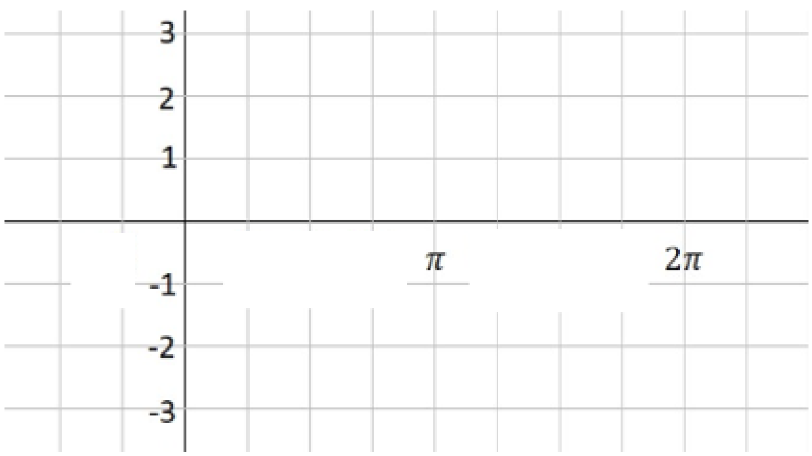
5.3 The Tangent Function

Let's track what happens to the values of $y = \tan\theta$ as θ, a standard-position angle, gets larger.



		$y = \tan\theta$
Q1	As θ increases from 0 to $\frac{\pi}{2}$	tangent values (y/x)
Q2	As θ increases from $\frac{\pi}{2}$ to π	tangent values (y/x)
Q3	As θ increases from π to $\frac{3\pi}{2}$	tangent values (y/x)
Q4	As θ increases from $\frac{3\pi}{2}$ to 2π	tangent values (y/x)

$y = \tan\,\theta$ (also often written as $y = \tan x$)

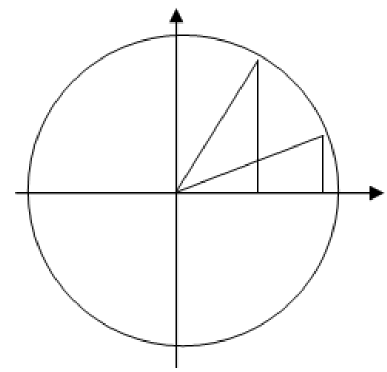


θ	$\tan\,\theta$
0	
$\frac{\pi}{4}$	
$\frac{2\pi}{4} = \frac{\pi}{2}$	
$\frac{3\pi}{4}$	
$\frac{4\pi}{4} = \pi$	
$\frac{5\pi}{4}$	
$\frac{6\pi}{4} = \frac{3\pi}{2}$	
$\frac{7\pi}{4}$	
$\frac{8\pi}{4} = 2\pi$	

Domain: x-intercepts: Asymptote equations:

Period: Range:

The tangent graph shows how the ***slope*** of the terminal arm of a standard-position angle θ changes, as the angle increases in size.

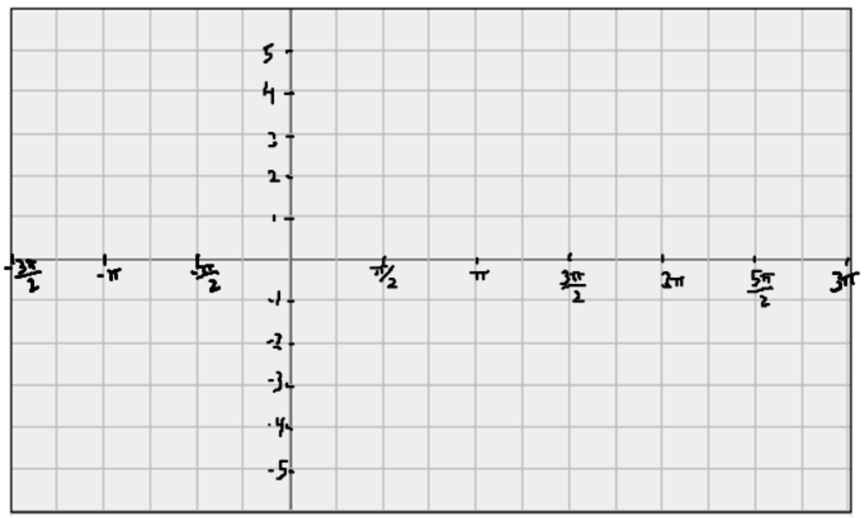


$$\tan\theta = \frac{y}{x}$$

$$slope = \frac{change\ in\ y}{change\ in\ x} = \frac{rise}{run}$$

Try

1) Sketch the graph of $y = \tan\theta$. Try to figure out points on the graph yourself (rather than just using your calculator), by using the slope idea discussed above.



Period:

x-intercepts:

Domain:

Asymptote equations:

2) For the graph of $y = \tan(3\theta)$

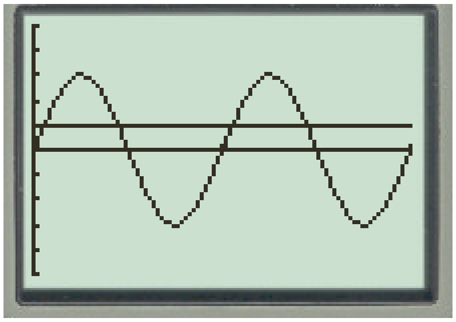
a) what does the “3” do?

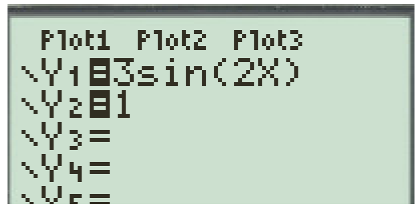
b) period of $y = \tan(3\theta)$

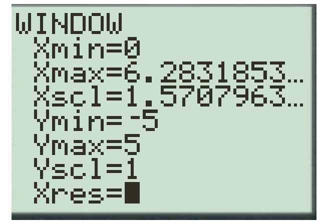
c) asymptote equations for $y = \tan(3\theta)$

5.4 Equations and Graphs of Trigonometric Functions

Below we see how we can solve a trigonometric equation, ***graphically***.







a) What is the equation that is being solved?

b) The window has been restricted to match the domain for this question. What is that domain?

c) How many solutions are there, in this domain? Mark them on the calculator graph screenshot, shown above.

Remember, there are two ways to solve equations GRAPHICALLY

Intersection Method

1) Enter the LHS of the equation as Y_1

2) Enter the RHS of the equation as Y_2

3) Set the *Xmin* and *Xmax* values using the given domain.

4) Use the "intersect" feature to find each place where the LHS = RHS.
The *x*-values of the intersections are the equation's solutions.

X-intercepts (zeroes) Method

1) Collect all terms of the equation on one side of the equals sign, so it looks like

$\boxed{} = 0.$

2) Enter the equation as Y_1

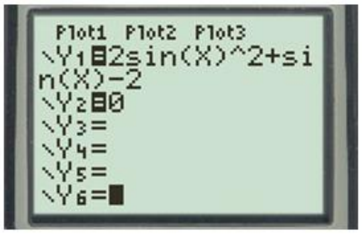
3) Set the *Xmin* and *Xmax* values using the given domain.

4) Use the "Zero" feature to find each *x*-intercept. These *x*-values are the equation's solutions.

Try

Solve ***graphically***, correct to 1 decimal place. Include a sketch of the graph with the solutions marked on it.

$$2\sin^2 x + \sin x - 2 = 0, \text{ for } 0^\circ \le x \le 720^\circ$$



Example

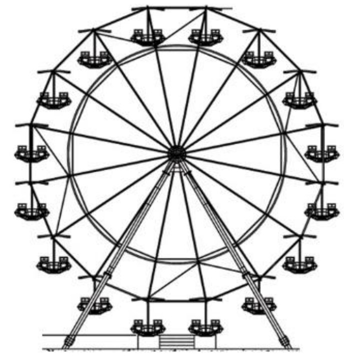
Consider the trigonometric equation $6\sin\left(\frac{\pi}{4}x\right)+8=10$

a) Solve ***graphically*** for $0 \le x < 2\pi$, correct to 4 decimal places. Include a sketch of the graph with the solutions marked on it.

b) Find the general solution, ***algebraically***, correct to 4 decimal places.

Example

Suppose the pictured Ferris wheel has diameter 40 meters, and the height of the seat where you first get on is 2 meters above the ground. This wheel takes 2 minutes to rotate and travels at a constant speed.



- Minimum height?
- Maximum height?
- Center line height?
- Period length in seconds?

a) Sketch a complete period of the graph, showing the height of a passenger above the ground as a function of time, in seconds. Give the coordinates of 5 key points.

t	h

b) Create a sinusoidal equation for this graph.

c) How high above the ground is a passenger 12 seconds after getting on, correct to one decimal place?

d) During the first rotation of the Ferris wheel, what is the first time that the passenger reaches a height of 30 meters above the ground? Solve this ***graphically***.

Chapter 6: Trigonometric Identities

6.1 Trigonometric Identities

In this chapter we talk about trigonometric ***identities***. Trigonometric identities look like trigonometric equations, but there's a difference.

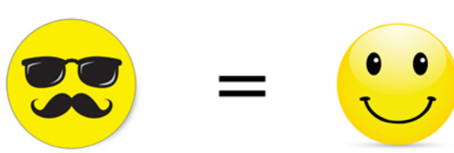
identities	equations
$2x+4=2(x+2)$	$2x+3=9$
$\dfrac{\left(x^2-9\right)}{x+3}=x-3$	$x^2+5x=-6$
$\csc x=\dfrac{1}{\sin x}$	$\sin x=0.53$

Identity - an equation that is true for **ALL** permissible values

When we are given an identity to prove, we see different expressions on the left-side and right-side of the equation. Proving the identity means we must ***change*** the expressions so that we end up with the SAME expression on both sides of the equation.

Our tools to do this are:

- algebra skills (getting common denominator, combining terms, factoring)
- basic identity substitutions



We will be verifying and proving trigonometric identities.

- ***Verifying*** an identity means we show it *seems* true. Done by:
  - *substituting* in a specific value and confirming that, for that value, the left and right sides of the identity are equal
  - *graphing* the left and right sides of the identity separately, and confirming that the graphs are exactly the same in that window

- ***Proving*** an identity means using algebra and/or Basic Identities to change the form of one or both sides of the identity, until the two sides are exactly the same.

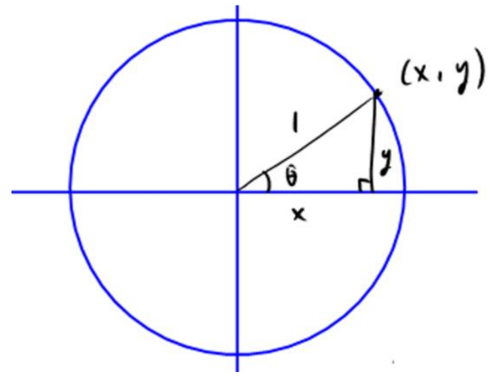
Example

Verify the given identity, for the value $x = \frac{\pi}{5}$ $\qquad \sec x = \frac{\tan x}{\sin x}$

Example

Verify graphically: $\qquad \frac{\sin^3 x}{\sin x} = \sin^2 x$

Basic Identities



Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Addition Identities

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Double Angle Identities

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta \qquad \tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\cos(2\theta) = 1 - 2\sin^2\theta$$

Practice Using Basic Identities

Match the expressions on the left with those on the right-hand column. Put the letter of the expression that matches in the blank provided. Each gets used exactly once.

________	1.	$\dfrac{\sin B}{\cos B}$	A.	1
________	2.	$\dfrac{1}{\cos B}$	B.	$\sin^2 B$
________	3.	$\csc^2 B$	C.	$\cot^2 B + 1$
________	4.	$\sin^2 B + \cos^2 B$	D.	$\cos^2 B$
________	5.	$\cot B \sin B$	E.	$1 + \tan^2 B$
________	6.	$\dfrac{\cos B}{\sin B}$	F.	$\cot B$
________	7.	$1 - \cos^2 B$	G.	$\tan B$
________	8.	$\dfrac{\cos^2 B}{1+\sin B}$	H.	$\sec B$
________	9.	$\sec^2 B$	I.	$\csc B$
________	10.	$\dfrac{1}{\sin B}$	J.	$\cos B$
________	11.	$\dfrac{\cos B}{\cot B}$	K.	$1 - \sin B$
________	12.	$1 - \sin^2 B$	L.	$\sin B$

More Practice

Simplify each expression below. Look for substitutions you can make, using basic identities. Your final answer should contain no more than one trigonometric function.

1. $\dfrac{\cos^2\theta}{\sin^2\theta}$

2. $\tan\theta\sec\theta\cos\theta$

3. $1-\cos^2\theta$

4. $\cos^2\theta-1$

5. $1+\tan^2\theta$

6. $\sin^2\theta+\cos^2\theta+1$

7. $\csc^2\theta-\cot^2\theta$

8. $\sin^2\theta+\cos^2\theta+\tan^2\theta$

9. $\dfrac{\sin^2\theta+\sin\theta}{\cos\theta+\cos\theta\sin\theta}$

10. $\dfrac{\sqrt{1-\sin^2 x}}{\sqrt{1+\tan^2 x}}$

11. $1-\sec^2 x$

12. $\sec^2 x-1$

6.0 Algebra Skills Used in Chapter 6

Multiplying Trigonometric Expressions

1. $\sin x(2\sin x - 1)$

2. $(\cos x + 2)(\cos x - 7)$

3. $(\cos x - 3)^2$

Factoring Trigonometric Expressions

Greatest Common Factor

1. $\sin^2 x - 3\sin x$

2. $5\tan^2 x + 15\tan x$

Difference of Perfect Squares

1. $\sin^2 x - 1$

2. $1 - \tan^2 x$

Trinomials

1. $2\cos^2 x + \cos x - 1$

2. $3\sin^2 x + 2\sin x - 1$

Adding/Subtracting Trigonometric Terms

We can only add ***like terms***

- Terms must contain the same angle
- Terms must use the same trigonometric function

Which of these terms can be combined?

$4\sin x + 3\cos x$ $\qquad$ $2\sin x + 5\sin 2x$

$3\sin x + 4\sin x$ $\qquad$ $2\sin^2 x + 4\sin x$

Errors to Avoid

Omitting the angle

$\cos + 4\cos$ $\qquad$ These terms contain no angle - they don't mean anything!

Incorrect cancelling

You can never "cancel" the angle, or any part of the angle, in a trigonometric expression.

$$\frac{\cos x}{x} \qquad \frac{\tan 2x}{2}$$

More incorrect cancelling

You can NEVER cancel just a portion of a factor.

$\frac{\cos x+1}{\cos x}=$ You **CANNOT** cancel the "$\cos x$" on the top with the "$\cos x$" on the bottom!

$\frac{(\cos x+1)(\cos x-1)}{(\cos x+1)}=$ You **CAN** cancel the "$\cos x+1$" factors

To CORRECTLY simplify a rational expression, factor it completely.
If the numerator and the denominator contain the same factor, you can reduce.

For example: $\frac{\sin^2 x+8\sin x+12}{\sin^2 x+\sin x-30}=$

Notice that we **canno**t reduce $\frac{\sin x+2}{\sin x-5}$ by canceling the $\sin x$'s, because $\sin x$ is NOT a factor of the numerator and the denominator. $\frac{\sin x+2}{\sin x-5}\neq\frac{2}{-5}$

Distributing when you can't:

$\cos(x+y)$ This does **NOT** equal $\cos x + \cos y$!
We are not multiplying "cos" with $(x + y)$.

What this expression DOES mean is the cosine of the angle "$x + y$"

For example, consider what happens if $x=15^\circ$ and $y=28^\circ$

$\cos\left(15^\circ+28^\circ\right)=$

$\cos\left(15^\circ\right)+\cos\left(28^\circ\right)=$

This shows us that $\cos(x+y)\neq\cos x+\cos y$

6.2 Sum, Difference, and Double-Angle Identities

Sum/Difference Identities

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$ $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$ $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Double Angle Identities

$\sin(2\theta)=2\sin\theta\cos\theta$ $\cos(2\theta)=\cos^2\theta-\sin^2\theta$ $\tan(2\theta)=\dfrac{2\tan\theta}{1-\tan^2\theta}$

$\cos(2\theta)=2\cos^2\theta-1$

$\cos(2\theta)=1-2\sin^2\theta$

Examples

Find the exact value of the following expressions.

a) $\cos 195^\circ =$

b) $\sin\left(\dfrac{5\pi}{12}\right)=$

c) $\sin 50^\circ \cos 10^\circ + \cos 50^\circ \sin 10^\circ =$

Write each expression in a simpler form, using identities. Give exact value, if possible.

a) $2\sin 15^\circ \cos 15^\circ =$

b) $\cos^2(\pi/8) - \sin^2(\pi/8) =$

c) $1 - 2\sin^2(8) =$

d) $10\cos^2(x) - 5 =$

Given that $\sin A = \frac{8}{17}$, where A is a Q1 angle; and $\cos B = \frac{24}{25}$, where B is a Q4 angle.

a) Draw two coordinate systems. Sketch a reference triangle for angle A on one of the systems, and one for angle B on the other.

b) Use identities to find the exact value of:

$\sin(A - B)$

$\tan(2A)$

$\sin 2B$

$\cos 2A$

6.3 Proving Identities

When we prove identities:

- Step by step, use algebra and/or Basic Identities to change the way either the left- hand side (LHS) or the right-hand side (RHS) looks.
- Think of the " = " sign separating the LHS and RHS as a barrier. Don't take terms from one side of the equals sign to the other.
- When the LHS and the RHS look exactly the same, the identity is proven.

Strategies for Proofs

- Write each step directly below the previous one.
- Don't skip steps – aim to be as CLEAR as possible.
- See if there's any factoring you can do, especially GCF or difference of squares.
- Don't cancel anything, unless you have identical factors on the top and bottom of an expression.
- If rational expressions are added /subtracted together, get a common denominator so you can combine the expressions and simplify.
- If possible, substitute known identities to simplify expressions.
- If the LHS and RHS look as below, where they are ***almost*** reciprocals of each other, multiply one side, top and bottom, by the conjugate of the binomial. Then use a Pythagorean identity to simplify further.

$$\frac{\cos y}{1-\sin y} = \frac{1+\sin y}{\cos y}$$

Try these strategies to prove the identities on the following pages.

1. $\left(\sin x+\cos x\right)^2 \qquad = \qquad 1+2\sin x\cos x$

2. $\tan^2 x\sin^2 x-\tan^2 x \qquad = \qquad -\sin^2 x$

3. $\sec^4 x \qquad = \qquad \tan^4 x+2\tan^2 x+1$

4. $\left(3-3\sin x\right)\left(3+3\sin x\right) \qquad = \qquad 9\cos^2 x$

5. $\dfrac{1}{1+\cos x}+\dfrac{1}{1-\cos x}$ $=$ $2\csc^2 x$

6. $\dfrac{\cos y}{1-\sin y}$ $=$ $\dfrac{1+\sin y}{\cos y}$

7. $2\sec x$ $=$ $\dfrac{\cos x}{1+\sin x}+\dfrac{1+\sin x}{\cos x}$

8. $\dfrac{\sin x}{1+\cos x}$ $=$ $\dfrac{1-\cos x}{\sin x}$

6.4 Solving Trigonometric Equations Using Identities

Some trigonometric equations cannot be solved until they are re-written in a different form, using trigonometric identities.

Example

Algebraically solve this equation, giving the ***general solution***, in radian measure.

$$2\sin x = 9 - 4\csc x$$

Example

Algebraically solve this equation

$\cos 2x + \cos x = -1$, for $0^\circ \le x < 360^\circ$

Try

Algebraically solve this equation

$$2\cos x + 1 - \sin^2 x = 3 \text{ , for } 0 \le x < 2\pi$$