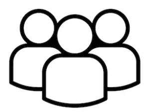
Chapter 7: Exponential Functions

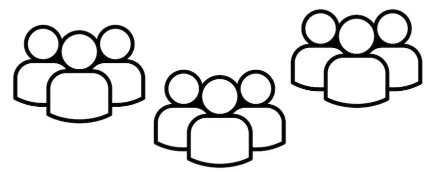
7.1 Characteristics of Exponential Functions

An exponential function is a function where the ***exponent*** includes a variable, and the ***base*** is larger than zero, not equal to 1. Exponential functions are used to model many real-life situations of change – such as population growth, radioactive decay and compound interest.

For example –
Suppose you greet three people.



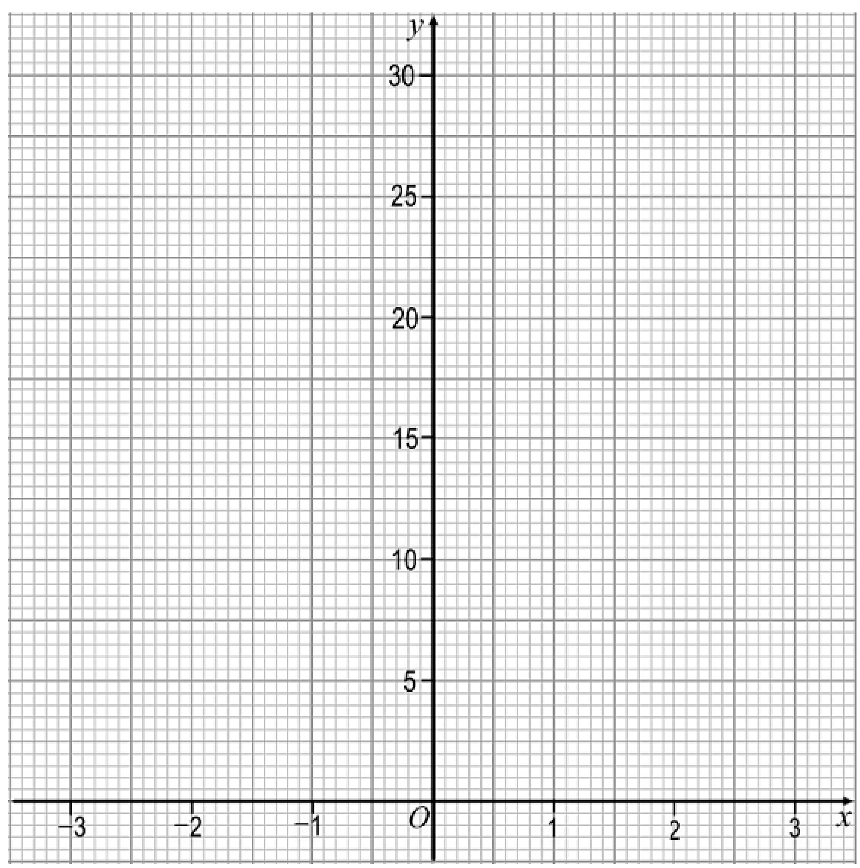
Each person you greeted goes on to greet 3 different people.



If this pattern continues, you can see that the number of people greeted grows very quickly.

Consider the function $y = 3^x$.
a) Complete the table, then sketch the graph of $y = 3^x$ on the grid.

x	y



b) State the graph's:
domain

range

y-intercept

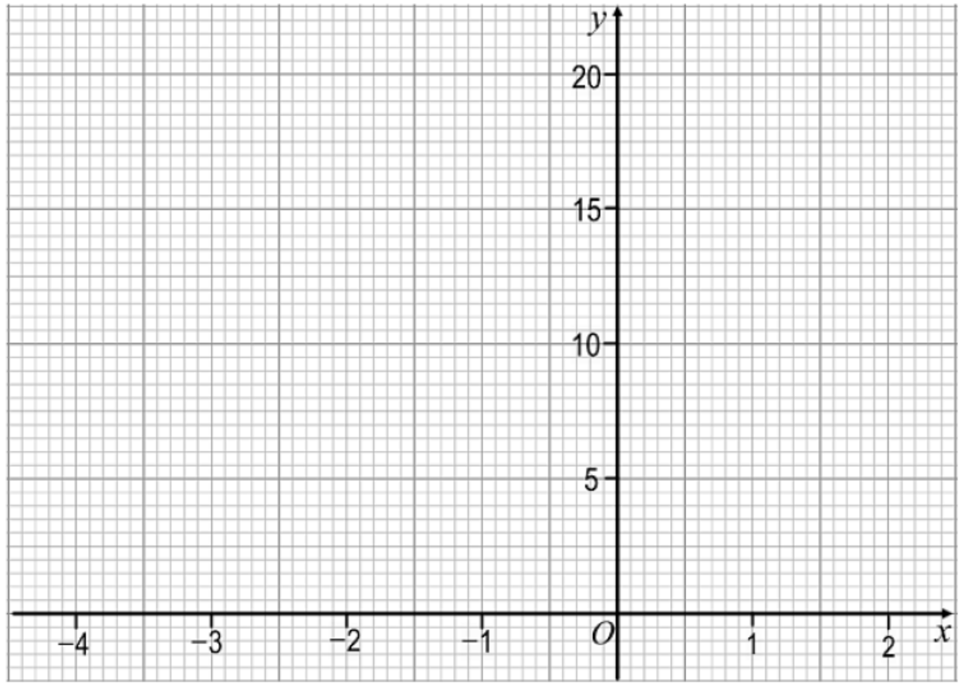
x-intercept

horizontal asymptote equation

Example

a) Create a table, then sketch the graph of the exponential function $y = \left(\frac{1}{2}\right)^x$ on the grid.

x	y



b) State the graph's:

domain

range

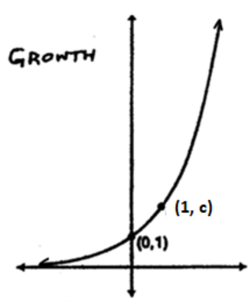
y-intercept

x-intercept

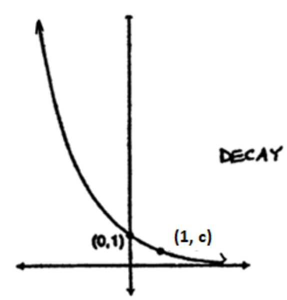
horizontal asymptote equation

$y = c^x$, when $c > 1$

$y = c^x$, when $0 < c < 1$



C > 1



0 < c < 1

7.2 Transformations of Exponential Functions

Predict what will happen to the graph of $y = 5^x$ when each of the following changes is made to the equation:

$y = 5^x + 1$ ____

$y = 5^{x-4}$ ____

$y = 5^{2x}$ ____

$y = 5^{3x-12}$ ____

$y = -2\left(5^x\right)$ ____

Horizontal stretch by a factor of $\frac{1}{b}$

Horizontal translation

Vertical stretch by a factor of a

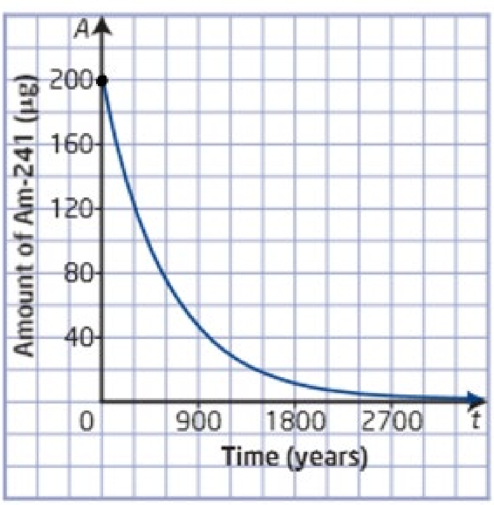
Vertical translation

$$y = a(c)^{b(x-h)} + k$$

If b < 0 then there is a reflection over the y-axis (horizontal reflection)
If a < 0 then there is a reflection over the x-axis (vertical reflection)

Creating Exponential Functions

The radioactive element americium is used in household smoke detectors. Am-241 has a half-life of approximately 432 years. The average smoke detector contains 200 μg of Am-241. What is the exponential function that models the graph showing the radioactive decay of 200 μg of Am-241? (TB, p 353)



<u>Create the Equation</u>

1) The population of a town triples every 6 years. Suppose that 4000 people lived in the town in the year 2006.
a) equation

b) How many people would be living in the town in 2050?

2) A bacterial culture doubles every 2 hours. This culture had 22 000 bacteria at time $t = 0$.
a) equation

b) How many bacteria would be in the culture after 5 hours?

3) The half-life of a radioactive sample is 4 hours. The sample size was originally 60 g.
a) equation

b) How many grams would be in the sample after 11 hours?

4) For every meter that you descend into water, 5% of light is blocked.
(If you start with 100% of the light and 5% is blocked, what percentage of light do you still have?)
a) equation

b) What percentage of light would still pass through the water, at a depth of 15 meters?

5) $5000 is invested at 7.2% compounded **<u>annually</u>**.
a) equation

b) How much money would you have after 3 years?

7.3 Solving Exponential Equations

Remember the rules for working with exponents:

$a^m a^n =$

$\left(a^m\right)^n =$

$\left(ab\right)^m =$

$\left(\dfrac{a}{b}\right)^m =$

$\dfrac{a^m}{a^n} =$

$a^{\frac{m}{n}} =$

Exponential equations are ones where the ***variable is in the exponent***. We can solve these equations by

- Writing the left side of the equation and the right side of the equation so they each use the same base.
- Then, we use the fact that if $a^x = a^y$, it forces $x = y$, to finish solving the equation.

Example

$$8^{4x-1} = \left(\frac{1}{2}\right)^{x+5}$$

To Try

1. Rewrite the expressions so they have the same base, then solve the equation.

a) $\left(\frac{1}{25}\right)^{4x} = (125)^{3x+2}$

b) $(8)^{2x+7} = 16^{4x+2}$

c) $16^{3x} = 8^{3x-1}64^{x}$

2.For how long does one need to invest \$2000 in an account that earns 6.1% compounded quarterly, before it increases in value to \$2500?

$$A = P(1+i)^n$$

P = principal amount deposited
i =interest rate per compounding period,
in decimal form
n = number of compounding periods

3. The population of a town triples every 6 years. If 4000 people lived there in 2009, how many will be in the town 2030? (Round down to the nearest whole person.)

Chapter 8: Logarithmic Functions

8.1 Understanding Logarithms

A logarithm tells how many copies of one number we need to multiply together, to create a different number.

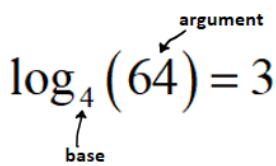
For example:
How many 4's do we have to multiply together to get 64?

$4 \times 4 \times 4 = 64$, which shows we have to multiply 3 of the "4's" to produce 64

This tells us the ***logarithm*** is 3.

Because $4 \times 4 \times 4 = 64$, we can say: $\log_4(64) = 3$

- we can read this as "The logarithm base 4 of 64 is equal to 3"
- we can shorten it a bit, and say "log base 4 of 64 equals 3"



A logarithm tells us how many copies of the BASE we need to multiply together, to create the ARGUMENT – in other words, the logarithm is the ***exponent*** we raise the base to, in order to produce the argument.

Try These

1. $\log_4(16) =$
2. $\log_3(27) =$
3. $\log_{0.5}(0.5) =$
4. $\log_a(a^7) =$
5. $\log_2(2^{-3}) =$
6. $\log_1 7 =$
7. $\log_5\left(\frac{1}{25}\right) =$
8. $\log_3 0 =$
9. $\log_2(-4) =$
10. $\log_6(\sqrt{6}) =$
11. $\log_2(\sqrt[3]{2^4}) =$
12. $\log_8(1) =$

For a logarithm to make sense, we need the argument and the base to obey these restrictions:

argument	**base**

Notation

log base 10 is called **COMMON log** — $\log_{10} x$ *is written* $\log x$

log base *e* is called **NATURAL log** — $\log_e x$ *is written* $\ln x$

The number *e* is a very important irrational number. Its decimal expansion starts out:

$$e \approx \mathbf{2.7182818284590452353602874713\ldots}$$

https://www.popularmechanics.com/science/math/a24383/mathematical-constant-e/

Changing Form

Exponents and logarithms are closely connected. Look at these two equations:

$\log_4 64 = 3$ and $4^3 = 64$

logarithmic form **exponential form**

Both equations show the relationship between the numbers 4, 3 and 64. We need to know how to change equations from one form to the other, as in some questions one form is better than the other.

To Try

1. Change form.

a) $\log_6 216 = 3$

b) $\log_p q = r$

c) $\log 1000 = 3$

d) $7^2 = 49$

e) $5^{x+y} = a$

f) $49^{1/2} = 7$

2. Solve for *x*.

a) $\log_2 (x-1) = 3$

b) $\log_6 x = -2$

c) $\log_x 8 = 3$

d) $\ln x = 2$

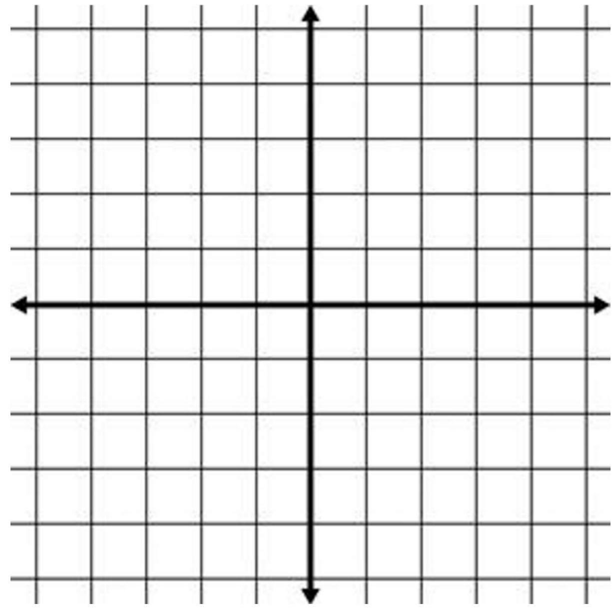
e) $\log_2 \left(\log_9 x\right) = -1$

f) $4^{\log_4 7} = x$

Graphing an Exponential Function and its Inverse

a) Fill in the table below, and sketch the graph of the exponential function, $y = 2^x$.

x	y



b) Identify the following:

domain

range

asymptote equation

x-intercept, if it exists

y-intercept, if it exists

c) Give the equation of the ***inverse*** of: $y = 2^x$. Inverse's equation is: ____________

d) For the equation of the inverse that you found in part c), complete the table at right and sketch the graph on the grid above.

x	y

e) For the inverse graph, what are its:

domain

range

asymptote equation

x-intercept, if it exists

y-intercept, if it exists

f) Rewrite the inverse equation from part c) in logarithmic form:

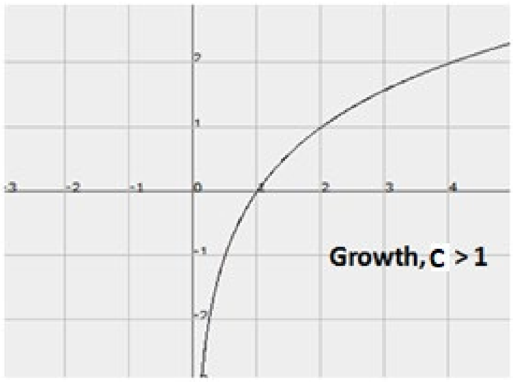
Conclusions:

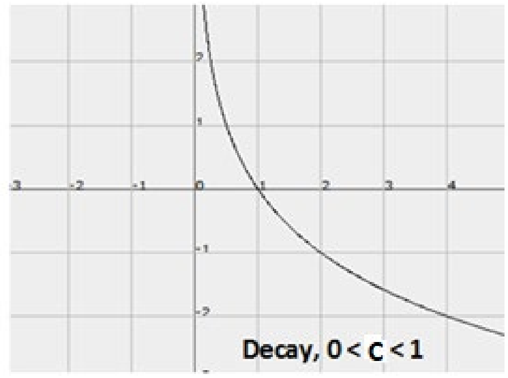
$\log_a\left(a^x\right) =$ $\qquad$ $a^{\log_a x} =$

8.2 Transformations of Logarithmic Functions

The graphs of logarithmic functions can be grouped into two categories:

- if the logarithm's base is larger than one, the graph is ***increasing***
- if the logarithm's base is between zero and one, the graph is ***decreasing***





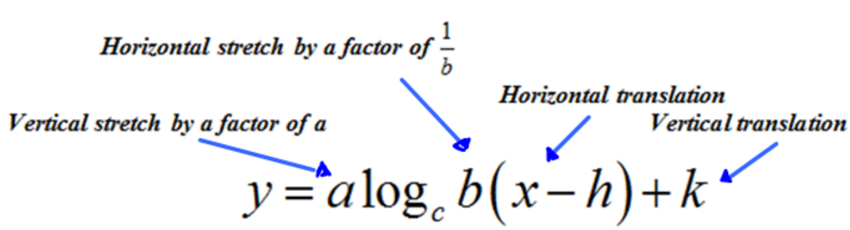
What characteristics are the ***same*** for all untransformed logarithmic graphs?

Predict what will happen to the graph of $y = \log_3 x$ when each of the following changes is made to the equation:

$y = \log_3 x - 5$ ____

$y = -4\log_3 x$ ____

$y = \log_3\left(-\frac{2}{5}(x+3)\right)$ ____



If b < 0 then there is a reflection over the y-axis (horizontal reflection)
If a < 0 then there is a reflection over the x-axis (vertical reflection)

8.3 Laws of Logarithms

Product Law:	$\log_c(MN) = \log_c M + \log_c N$
Quotient Law:	$\log_c\left(\dfrac{M}{N}\right) = \log_c M - \log_c N$
Power Law:	$\log_c\left(M^P\right) = P\log_c M$

To Try:

1. Evaluate without using use the "log" button: $\log_3 54 - \log_3 2 =$

2. Find the value of each of the following without using a calculator:
   a) $\ln 1$ b $\ln e$ c) $\ln e^4$

3. Evaluate without using the "log" button: $\log_{14} 4 + \log_{14} 49 =$

Change of Base Formula: $\log_C A = ?$

1. Evaluate. Give answer correct to 4 decimal places. $\log_2 18 =$

2. Express as a single logarithm. $\dfrac{\log 30}{\log 5}$

3. Rewrite this equation so you can graph it on a graphing calculator: $y = \log_4 x$

1. $\log_3\left(4y^2\right)$	6. $\frac{1}{2}(\log b - \log c)$
2. $2\log_4 b + 3\log_4 c$	7. $\log\left(\frac{\sqrt{a}}{c^2}\right)$
3. $\ln(ab)$	8. $\log\left(x^2 y\right)^4$
4. $\log\left(\frac{a}{b}\right)$	9. $3\log x - \log w^2$
5. $\frac{1}{2}\log a + 2\log c$	10. $\log\left(\frac{1000a^2}{c}\right)$

11. $\log_5\left(5x\sqrt{y}\right)$	16. $\log_7 y - 2\log_7 w + \log_7(5x)$
12. $\log\left(\frac{\sqrt{bc}}{a}\right)$	17. $\log a + 3\log b - 2\log c$
13. $2\log a - 4\log b$	18. $5\log_4 2 - \frac{1}{3}\log_4 8$
14. $\log\left(a^2c\right)$	19. $\frac{\log_5 x}{4} - \log_5(3x)$
15. $\log\left(\frac{x}{yw}\right)$	20. $2\log c - (3\log a + \log b)$

8.4 Logarithmic and Exponential Equations

Solving Logarithmic Equations

1. Use logarithm laws to simplify equation into one of two forms:
   - **log $_c$ (argument) = number**
     - in this case, change to exponential form and solve
   - **log $_c$ (argument) = log $_c$ (another argument)**
     - in this case, set the two arguments equal

2. Use algebra to solve the equation you created in step 1.

3. Substitute each solution into the original equation. If the solution makes the argument become zero or a negative number, then it is an ***extraneous solution*** and must be rejected.

To Try:
Solve for *x*. Reject any extraneous solutions.

1. $\log_9 5 + \log_9 x = \log_9 30$

2. $\ln x + \ln 5 = 2$

3. $\ln 512 - \ln 8 = 3\ln x$

4. $\log_2(x-6) = 3 - \log_2(x-4)$

5. $2\log_4(x+4) - \log_4(x+12) = 1$

6. $\log_{12}(3-x) + \log_{12}(2-x) = 1$

Solving Exponential Equations with Different Bases

In chapter 7 we solved exponential equations by making each side of the equation use the same base. When that is hard to do, we can instead solve by taking the logarithm of each side of the original equation and solving the resulting equation.

To Try:
Solve for x.

1. $3^x = 2800$

2. $e^x = 2$

3. $3\left(4^{2x+3}\right) = 8^{4x-2}$

Earthquakes, Sound, pH

Logarithms can be used to solve applications comparing the intensity of earthquakes, the intensity of sounds, and the acidity or alkalinity of solutions. The Richter scale for earthquakes, the decibel scale for sounds and the pH scale for solutions are all base 10.

$$I = I_0 (10)^{R-r}$$

where:

I = intensity of a stronger earthquake
I_o = intensity of weaker earthquake

R = Richter magnitude of stronger earthquake
r = Richter magnitude of weaker earthquake

$$I = I_0 (10)^{(D-d)/10}$$

where:

I = intensity of a louder sound
I_o = intensity of a softer sound

D = decibel level of louder sound
d = decibel level of softer sound

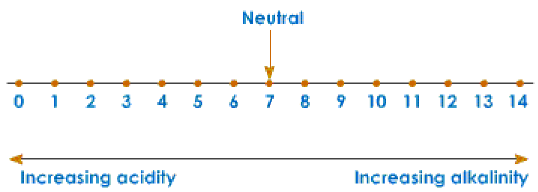
$$I = I_0 (10)^{P-p}$$

where:

I = new solution that is compared to original one
I_o = "original" solution

P = larger pH reading
p = smaller pH reading

- A neutral solution has a pH of 7.
- Solutions with pH larger than 7 are basic, or alkaline.
- Solutions with pH smaller than 7 are acidic.



$pH = -\log\left[H^+\right]$ where $\left[H^+\right]$ is the hydrogen ion concentration in moles per liter

The exponent is always a difference: larger reading – smaller reading
For sound questions, divide each decibel reading by 10.

Example

1a) In 1983, an earthquake measuring 5.5 on the Richter scale occurred in Columbia. In 1989, the San Francisco earthquake measured 6.9 on the Richter scale. How much more intense was the San Francisco earthquake than the Columbia earthquake?

b) Calculate the magnitude of an earthquake that is 1500 times as intense as the Columbia earthquake.

2. How much louder is a sound with an intensity of 112 dB compared to a sound with an intensity of 90 dB

b) If three different jets are flying together at an air show, each with a sound level of 120 decibels, then find the approximate total decibel level.

Example pH $= -\log\left[H^+\right]$ $I = I_0 (10)^{P-p}$

a) A beaker of acid has a hydrogen ion concentration of 3.5×10^{-6} mol/L. Calculate the pH of the acid.

b) Solution A has a pH of 5.7. Solution B is 1260 times more **acidic** than Solution A. Find the pH of Solution B.

For you to try . . .

1. How many times more intense is an earthquake with magnitude 8.3 than one with magnitude 6.7? (Round to nearest whole number.)

2. Bob was in an earthquake of magnitude 7.1. This earthquake was 420 times more intense than a smaller earthquake that his friend Joan was in. Find the magnitude of the smaller earthquake, correct to one decimal place.

3. How many times more intense is the sound of a power saw, 120 dB, than that of a leaf rustling, 10 dB?

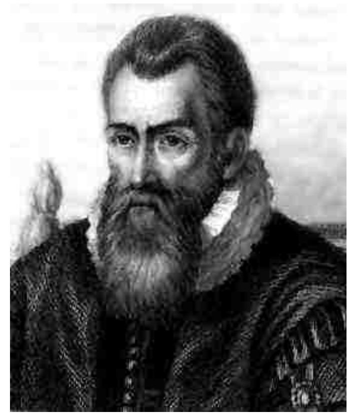
4. Two telephones in a home ring at the same time with a loudness of 80 decibels each. What is the decibel rating of the total loudness? (Note that 150d B is the sound of a jet engine, from 20 meters away, so the correct answer to this question is NOT 160 dB.)

5. Determine the pH of a solution, to the nearest tenth, if they hydrogen ion concentration is 3.4×10^{-4} mol/L.

6. Swimming pool water has a pH of 7.5. Sea water is about 8 times as alkaline as swimming pool water. What is the pH reading for sea water?

Answers:
1. The magnitude 8.3 earthquake is about ***40 times more intense*** than the 6.7 earthquake.
2. Magnitude of the smaller earthquake is 4.5 on Richter scale.
3. The sound of the power saw is about 10^{11} times as intense as that of a leaf rustling.
4. The total loudness is about 83 dB.
5. The solution has a pH of 3.5
6. The pH reading for sea water is about 8.4

Math History



John Napier lived from 1550-1617. He developed logarithms. In those days, logarithms were used mostly to do calculations. By using the laws of logarithms many difficult calculations could be simplified – instead of multiplying, one could use logarithms and then add, or instead of dividing, one could subtract. Here's an example:

Suppose you need to divide 217.39 by 25.461.

Logarithms helped like this:

$$\log_{10}\left(\frac{217.39}{25.461}\right) = \log 217.39 - \log 25.461$$
$$\log_{10}(quotient) = 2.337239563 - 1.405875457$$
$$\log_{10}(quotient) = 0.931364106$$
$$10^{0.931364106} = \text{quotient}$$
$$8.53815640 = \text{quotient}$$

The famous mathematician, Leonhard Euler, studied the number "*e*"

Leonhard Euler lived from 1707-1783. He published 530 books and papers during his lifetime. For the last 16 years of his life he was totally blind, but thanks to his phenomenal memory and ability to concentrate, he continued to generate a lot of mathematics. He would write formulas in chalk on a large slate for his secretary to copy down. He standardized these notations that you may know:

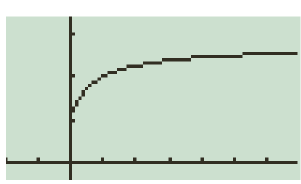
$f(x)$ for function notation
i for the imaginary unit, $\sqrt{-1}$

He came up with this formula, $e^{\pi i} + 1 = 0$, relating five of the most important numbers in mathematics.

The equation below tells us that as x gets huge, the y-values of the graph get closer and closer to the value of e:

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

Notice that the graph is approaching a horizontal asymptote which is somewhere between 2 and 3. The table of values shows that the y values are getting closer to the actual value of e.



X	Y1
150	2.7093
1150	2.7171
2150	2.7176
3150	2.7179
4150	2.718
5150	2.718
6150	2.7181