

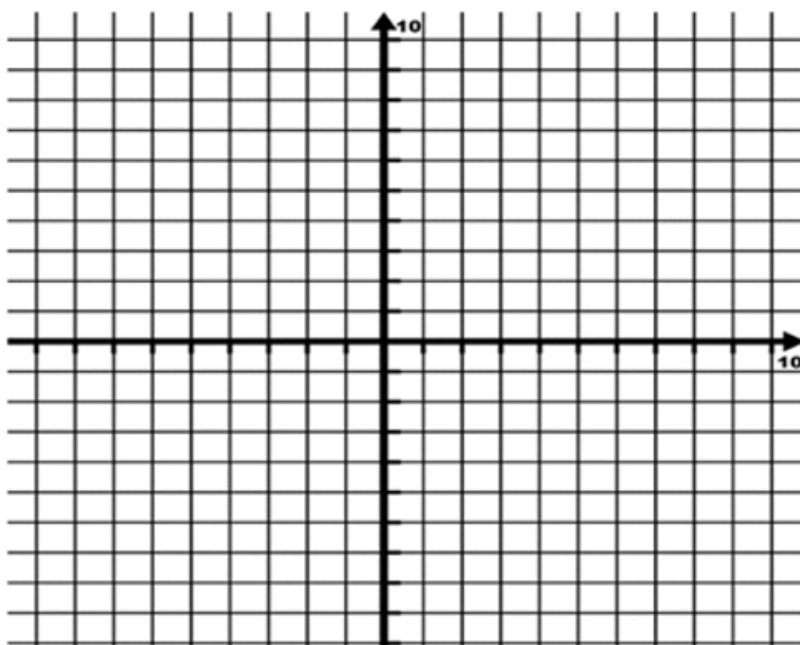
Chapter 9: Rational Functions

9.1 Exploring Rational Functions Using Transformations

Rational function are functions that can be written in the form $y = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.

Rational Functions with Linear Numerators and Denominators

Base rational function, $y = \frac{1}{x}$

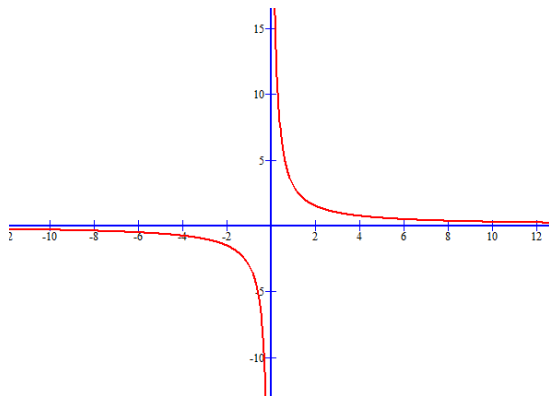


$y = \frac{1}{x}$	
x	y

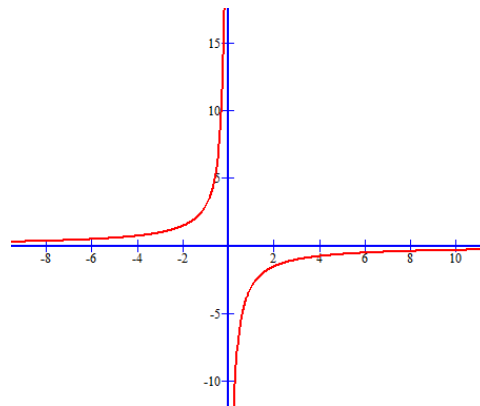
Non-permissible value (NPV)	
End behavior As $ x $ becomes very large, what does y do?	
Domain	
Range	
Equation of vertical asymptote	
Equation of horizontal asymptote	

In general, these are the *simplest* rational function equations, together with their graphs:

$$y = \frac{a}{x}, \quad a > 0$$



$$y = \frac{a}{x}, \quad a < 0$$



Transformations of $y = \frac{1}{x}$

For $y = \frac{a}{x-h} + k$, complete the table below.

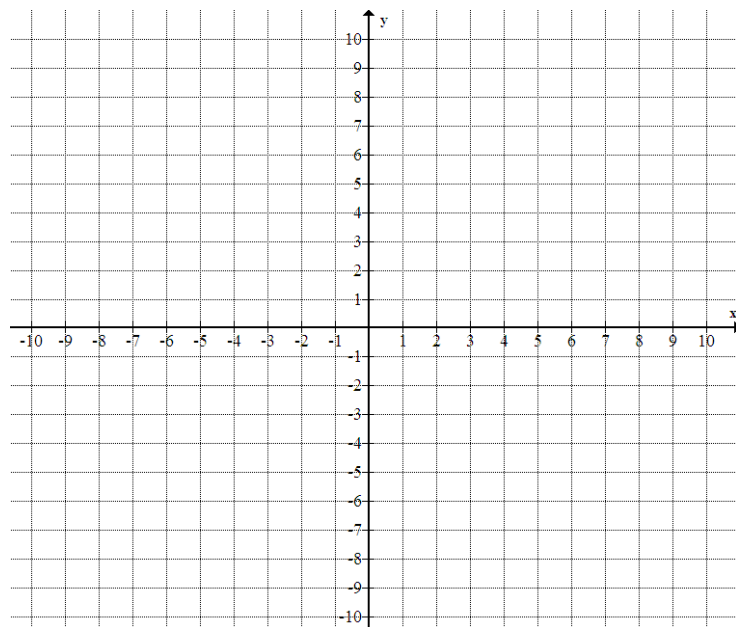
Non-permissible value (NPV)	
End behavior As $ x $ becomes very large, what does y do?	
Domain	
Range	
Equation of vertical asymptote	
Equation of horizontal asymptote	

To Try

1. Given the original rational function $y = \frac{1}{x}$ and the transformed function, $y = \frac{4}{x+3} - 5$:

a) Complete the tables below. For the first table, give 6 points found on the graph of the **original function** $y = \frac{1}{x}$. In the final table, give the image points that result after the transformations have occurred. Write the mapping notation in the heading of that table.

x	y
-2	
-1	
$-\frac{1}{2}$	
$\frac{1}{2}$	
1	
2	

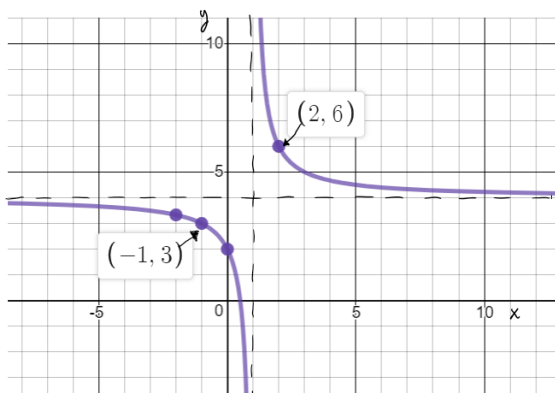


b) Accurately sketch the final transformed function. Include its horizontal and vertical asymptotes, drawn with dotted lines.

c) Give the equations of the asymptotes.

d) Use algebra to find the coordinates of the final graph's x -intercept and y -intercept.

2. Find the equation of the rational function graphed below.



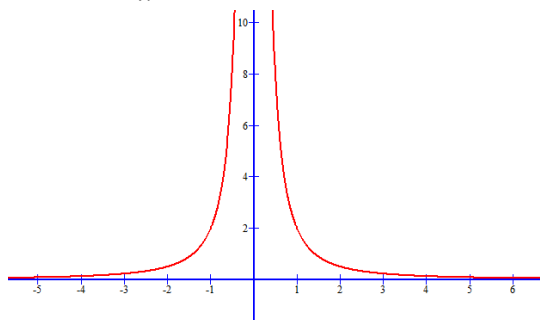
Example

Given the rational function $y = \frac{-4x + 3}{x + 2}$

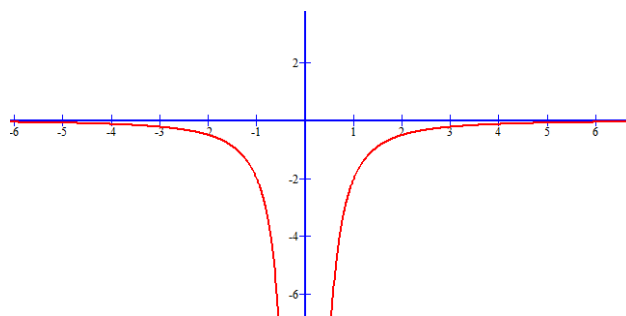
Find its NPVs, intercepts, and asymptote equations.

Rational Functions with Quadratic Term in Denominator, no other x's

$$y = \frac{a}{x^2}, \quad a > 0$$



$$y = \frac{a}{x^2}, \quad a < 0$$



Non-permissible value (NPV)	
Domain	
Range	
Equation of vertical asymptote	
Equation of horizontal asymptote	

Transformations of $y = \frac{1}{x^2}$

For $y = \frac{a}{(x-h)^2} + k$, complete the table below.

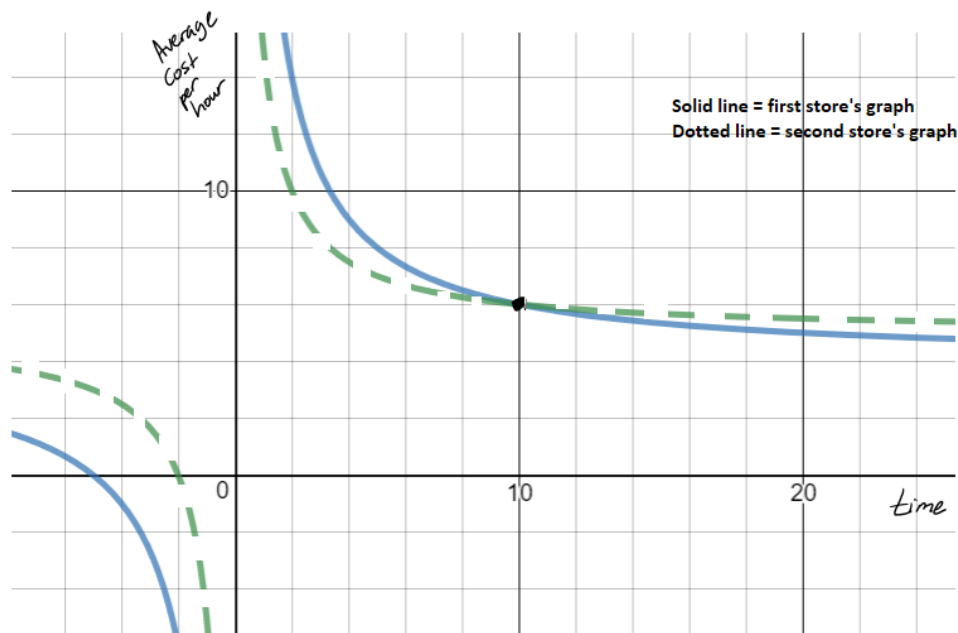
Non-permissible value (NPV)	
Domain	
Range	
Equation of vertical asymptote	
Equation of horizontal asymptote	

Example (TB p 445, #18)

Two stores rent bikes. The first store charges a fixed fee of \$20 plus \$4/h, and the second store charges a fixed fee of \$10 plus \$5/h.

a) Write equations for the average cost per hour for each store as a function of the rental time in hours.

b) Shown below are the graphs of the two equations from part (a). Which store is the better choice? Explain.



9.2 Analyzing Rational Functions

Some rational function equations are more complicated. To analyze and graph them, we *factor and simplify* their equations.

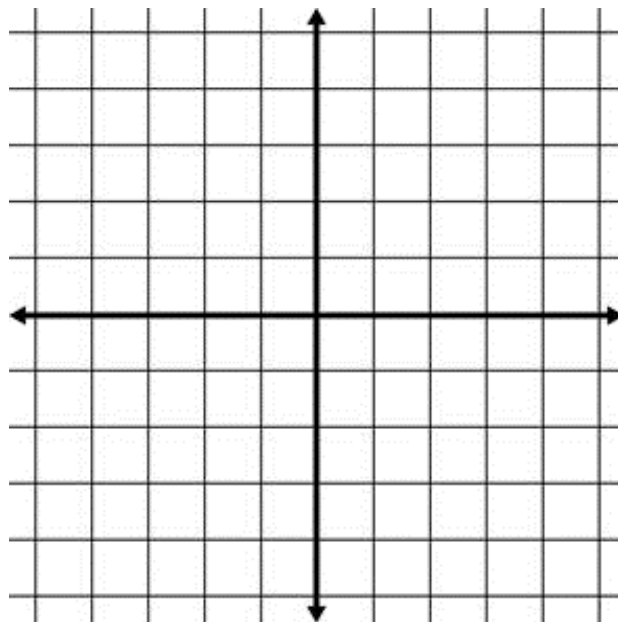
Example

Consider the rational function: $f(x) = \frac{x^2 + 7x + 12}{x + 4}$

a) Factor and simplify the function's equation.

b) NPV(non-permissible value) =
How does the graph behave near its NPV?

x $f(x)$
—
|



Point of Discontinuity (POD) – an ordered pair where the graph of a function does not exist. It occurs whenever the equation's numerator and denominator have a common factor that includes a variable.

Example

a) Complete the table, with the characteristics of the two graphs.

	$f(x) = \frac{x^2 + 2x - 8}{x - 2}$	$g(x) = \frac{x^2 + 6x + 8}{x - 2}$
Non-permissible value(s)		
Simplified form of equation		
Coordinates of x - and y -intercepts		

b) Graph these rational functions (same as the ones above) using technology. Below each equation draw a rough sketch of its graph.

$$f(x) = \frac{x^2 + 2x - 8}{x - 2}$$

$$g(x) = \frac{x^2 + 6x + 8}{x - 2}$$

When does a rational function have

- ***a point of discontinuity***
- ***a vertical asymptote?***

Key Ideas for Rational Function Graphs

1) Horizontal Asymptotes

Find the degree of the numerator and denominator.

Numerator degree < Denominator degree <i>horizontal asymptote equation:</i> $y = 0$
Numerator degree = Denominator degree <i>horizontal asymptote equation:</i> $y = \frac{\text{leading coefficient of num}}{\text{leading coefficient of denom}}$
Numerator degree > Denominator degree Graph will have a <i>slant asymptote</i>

2) NPVs, PODs, and vertical asymptotes

Factor numerator and denominator completely.

- Set *each factor of the denominator* = 0, to get all NPVs.
 - Is there a factor that cancels with a factor in the numerator? It gives the x -value of a POD.
 - Is there a factor that doesn't cancel with a numerator factor? It gives the location of a vertical asymptote.

3) Intercepts

- **y -intercepts** – substitute $x = 0$ into the function (either the original or the simplified form) and solve for y
- **x -intercepts** - set each factor of the simplified numerator = 0 and solve for x

4) Sketch

- Plot all x -intercepts and y -intercepts
- Show points of discontinuity (PODs) as “holes”, using an open circle
- Show all asymptotes as dotted lines.
- Find more points on the graph, as needed, by substituting into its equation.
- Make sure graph does not cross any vertical asymptotes.

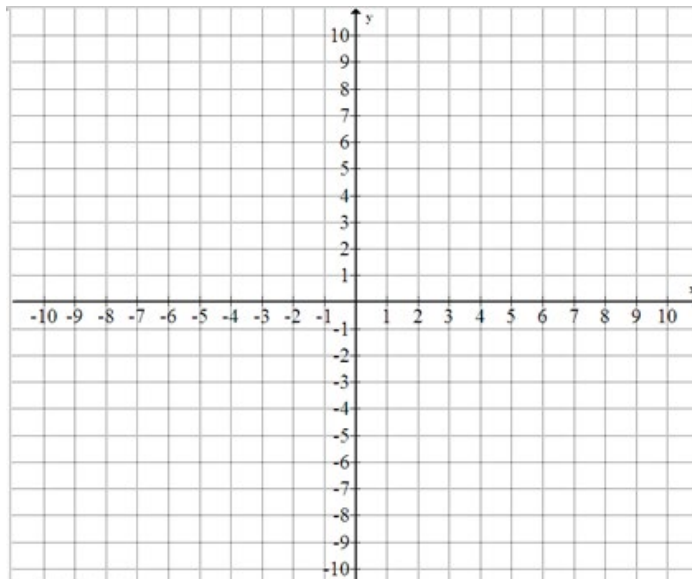
To try:

Original Equation	Factored form of equation	List all NPVs, and for each one identify if it gives a POD or a vertical asymptote. - Find the (x, y) coordinates of each POD . - Find the equation of each vertical asymptote.	Horizontal asymptote equation or say “Slant”
$y = \frac{2x+10}{x^2+2x-15}$			
$y = \frac{2x^2+7x+6}{x^2-2x-8}$			
$y = \frac{x^2+3x-4}{x-1}$			

Without using technology, accurately sketch the function's graph: $y = \frac{x^2 + 3x - 10}{x^2 - 4}$

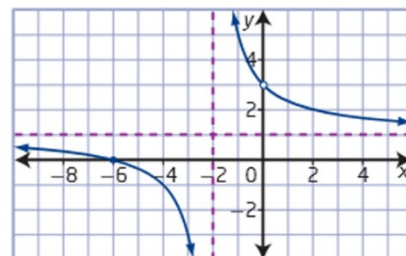
Give the values of the graph's:

- NPVs
- asymptote equations
- coordinates of PODs
- x - and y -intercepts



Example (TB page 453, #7a)

Write the equation of the pictured rational function.



9.3 Connecting Graphs and Rational Equations

To solve rational equations algebraically:

- Determine the value of all non-permissible values. List them.
- Find the least-common denominator (LCD).
- Multiply each term in the equation by the LCD, to eliminate fractions
- Solve this simpler equation. If a solution is an NPV, reject it.

Example

a) Solve algebraically:

$$\frac{3}{x} = 1 + \frac{x-13}{6}$$

b) Verify the solution graphically.

There are two ways to solve graphically.

1) Graph $Y_1 =$ LHS of equation

Graph $Y_2 =$ RHS of equation.

Find the x -values where the 2 graphs intersect.

OR

2) Collect all terms of the original equation on one side of the equals sign.

Graph this equation.

Find all of this graph's x -intercepts (zeroes)

To try:

1a) Find the roots of this rational equation, algebraically:

$$x + \frac{6}{x+2} - 5 = 0$$

b) Verify, graphically.

2a) Find the roots of this rational equation, algebraically:

$$1 + \frac{2}{x} = \frac{x}{x+3}$$

b) Verify the solution graphically.

Example

Bert has scored 7/10 on each of five math quizzes so far this year. He really wants an “A” in his quiz bin. He is sure he can get 10/10 on every quiz for the rest of the semester. How many more quizzes does he need to write in order to get 86% in his quiz bin?

Right now, Bert's quiz percentage = $\frac{\text{Bert's quiz pts}}{\text{total quiz pts}} =$

If x = number of additional quizzes written, and Bert succeeds in getting 10/10 on each quiz, then

Bert's new quiz percentage =

G.1 Geometric Sequences and Series

A sequence of numbers is a list of numbers in a specific order. They are referred to as the *terms* in the sequence. What is the next term for each sequence below?

1, 2, 4, 8, 16, 32, 64 . . .
5, 10, 15, 20, 25, 30 . . .

1, 4, 9, 16, 25, 36, 49 . . .
1, 1, 2, 3, 5, 8, 13, 21 . . .

There are many types of sequences. The ones shown below are all the same kind. What do they have in common?

6, 12, 24, 48. . .

-5, 15, -45, 135...

16, 4, 1, $\frac{1}{4}$, $\frac{1}{16}$...

-3, -12, -48, -192...

Geometric sequences are created by multiplying each term after the first one by a constant number which is called the *common ratio*. In formulas, they use the letter *r* to stand for the common ratio.

How can you figure out the value of *r*?

We use the notation t_n to identify terms in a sequence.

In the sequence beginning 3, 6, 12, 24, 48 . . .

$t_3 = 12$, because 12 is the third term of the sequence

$t_5 = 48$, because 48 is the fifth term of the sequence

The **first term** in the sequence could be referred to correctly as t_1 , but often we refer to it as "*a*". In the sequence 3, 6, 12, 24, 48 . . . we say $a = 3$.

To try:

Find these terms, for the sequence that begins: 3, 6, 12, 24, 48 . . .

a) $t_6 =$

b) $t_8 =$

c) $t_{26} =$

For any geometric sequence the n^{th} term, t_n , is given by the formula:

$$t_n = ar^{n-1} \quad \text{where } a = \text{first term}$$

$r = \text{common ratio, } r \neq 0$
 $n = \text{the term number}$

To try:

1. Which sequences are geometric? If a sequence is geometric, state its common ratio, r , and give the next 3 terms of the sequence.

a) 5, 10, 15, 20 . . .

b) 2, -4, 8, -16, 32...

c) 8, 2, $\frac{1}{2}$, . . .

d) x , $5x^2$, $25x^3$...

2. For each geometric sequence, find the requested value.

a) 8, 2, $\frac{1}{2}$, . . . Write the defining statement for this sequence in the form $t_n = ar^{n-1}$

b) 20, 10, 5 . . . find t_6

c) 21, -42, 84 . . . find t_{10}

d) $2x^2$, $4x^3$, $8x^4$... find t_9 .

3a) Consider the geometric sequence 2, -6, 18, -54, ...
13 122 is a term in this sequence. What term number is it?

b) Consider the geometric sequence 1.25, 5, 20...
327 680 is a term in this sequence. What term number is it?

4a) In a geometric sequence, $t_5 = 324$ and $t_8 = 8748$. Find the first two terms.

b) In a geometric sequence, $t_2 = 18$ and $t_5 = 2.25$. Determine the value of t_9 .

6. Between the Canadian censuses in 2001 and 2006, the number of people who could speak in Cree had increased by 7%. In 2006, 87 285 people could converse in Cree. Assume the 5-year increase continues to be 7%. To the nearest hundred, how many people will be able to converse in Cree in 2036?

G. 2 Geometric Series

A chessboard has 64 squares on it. Legend has it that the inventor of the game asked for the following reward from the King: “One grain of wheat on the first square, two on the second, 4 on the third, and so on, doubling the amount on every square until the board is complete.”



How many grains of wheat did the inventor ask for?

A *geometric series* is the **sum** of the first n terms of a geometric sequence, written as S_n .

S_3 - is the sum of the first 3 terms of a geometric sequence: $S_3 = t_1 + t_2 + t_3$

For the sequence above, $S_3 = 1 + 2 + 4 = 7$

S_4 - is the sum of the first 4 terms of a geometric sequence: $S_4 = t_1 + t_2 + t_3 + t_4$

For the chessboard sequence, $S_4 = 1 + 2 + 4 + 8 = 15$

For this question, we need to know the value of S_{64}

Here’s a way to find the value of a sum of n terms for any geometric series, S_n

$$\begin{aligned} S_n &= a + ar + ar^2 + ar^3 + \dots + ar^{n-2} + ar^{n-1} \\ -rS_n &= \quad ar + ar^2 + ar^3 + \dots + ar^{n-2} + ar^{n-1} + ar^n \end{aligned}$$

Subtracting, we get: $S_n - rS_n = a - ar^n$

Factor: $S_n(1 - r) = a(1 - r^n)$

Divide: $S_n = \frac{a(1 - r^n)}{1 - r}$

For any geometric sequence, the sum of the first n terms is found using:

$$S_n = \frac{a(1-r^n)}{1-r} \quad a = \text{first term}$$

r = common ratio, $r \neq 1$

n = number of terms to add up

Another form of the formula: $S_n = \frac{a-lr}{1-r}$ l = the last term

To Try:

1. Given the geometric series: $4 + 12 + 36 + \dots$ find the sum of the first 9 terms.

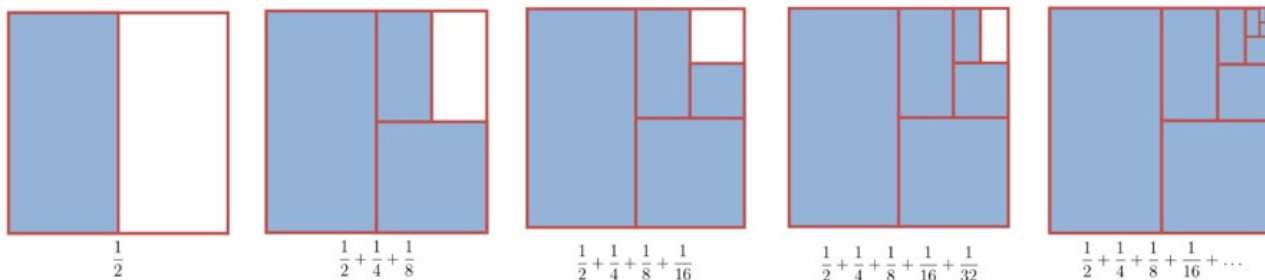
2. The sum of the first 12 terms of a geometric series is 24 570. The common ratio is -2 . What is the first term?

- 3a) Find the sum of this series: $0.025 + 0.075 + \dots + 54.675$

- b) Use an algebraic method to determine how many terms there are in the above sum.

4. An oil well produced 10 000 barrels of oil in the first month of production. Each month production is reduced by 10%. What is the total amount of barrels of oil produced by this oil well in its first two years of production?

G.3 Infinite Geometric Series



Consider the series illustrated above: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

- What do you notice about the terms?
- What do you think the sum is?

An *infinite geometric series* is one with an infinite number of terms – there is no last term.

With an infinite series, we will find that either:

Successive terms get smaller and smaller and smaller.

- The common ratio, r , has a value between -1 and 1 .
- This type of series is called ***convergent***.
- We **CAN** find the sum by using the formula: $S = \frac{a}{1-r}$

Successive terms stay the same size, or continue to get larger and larger

- The common ratio, r , satisfies either $r \geq 1$ or $r \leq -1$.
- This type of series is called ***divergent***.
- We **CANNOT** find the sum of this series. We say: “No finite sum.”

A geometric series has a **finite sum**, or **converges**, only when $-1 < r < 1$

$$S = \frac{a}{1-r}, \text{ only true if } -1 < r < 1$$

If you have an infinite geometric series where $r \geq 1$ or $r \leq -1$, it will not have a finite sum, it will **diverge**.

To Try:

1. Determine whether each infinite geometric series has a finite sum. If it does have a finite sum, find its value.

a) $10 - \frac{20}{3} + \frac{40}{9} - \dots$

b) $0.1 + 0.3 + 0.9 + \dots$

c) $32 + 8 + 2 + 0.5 + \dots$

d) $-1 - \frac{3}{4} - \frac{9}{16} - \frac{27}{64} \dots$

2a) The sum of an infinite geometric series is 63 and the first term is 21. Find the common ratio.

b) The sum of an infinite geometric series is $24/7$ and $r = -\frac{3}{4}$. Find the first term, a .

c) An infinite geometric series has the terms $a = 32$ and $t_4 = 4$. Determine the sum of the infinite series.

4. A ball is dropped from a height of 15 meters and bounces to 60% of the previous height. How far has the ball traveled when it **comes to rest**?

G. 4 Sigma Notation

Series are sums. We have been writing them in an expanded form, like this example:

$$32 + 8 + 2 + 0.5$$

Here is a different way to write the same series: $\sum_{n=3}^6 32(0.25)^{n-3}$

This notation, ***sigma notation***, uses the upper-case Greek letter sigma, Σ , and means we need to “SUM UP” all the terms in the series generated by the expression after the sigma. Sigma notation is used in many mathematics and science resources, so it’s useful to understand how it works.

To ***EXPAND*** a series written in sigma notation

- Substitute the bottom number – the one written below the sigma – into the expression. This gives the first term of the series.
- Keep substituting in consecutive numbers to get the next terms of the series.
- The last number to substitute in is the one written above the sigma. This gives the last term of the series.

$$\sum_{n=3}^6 32(0.25)^{n-3} = 32(0.25)^{3-3} + 32(0.25)^{4-3} + 32(0.25)^{5-3} + 32(0.25)^{6-3}$$

To Try:

1. Expand and evaluate: $\sum_{n=2}^4 3\left(\frac{1}{2}\right)^{n-1}$

2. Evaluate: $\sum_{k=5}^{21} 4(2)^{k-2}$

To **WRITE** a geometric series in sigma notation

- Figure out the values of a and r
- Write a general expression for the series, using the format: $t_n = ar^{n-1}$
- Determine how many terms you have, n . Use this for the number above the sigma.

Example

Write the given series in sigma notation: $5 + 15 + 45 + 135 + 405 + 1215 + 3645$

To Try

1a) Write the series: $3 + 9 + 27 + 81 + 243$ in sigma notation.

b) Write the series: $20 + 40 + 80 + \dots + 163\,840$ in sigma notation.

2. Determine each sum.

a) $\sum_{k=4}^{12} 8\left(\frac{1}{2}\right)^{k-2}$

b) $\sum_{n=7}^{17} (2)^n$

2. Find the sum of these series, correct to two decimal places.

a) $\sum_{k=1}^{\infty} 100(0.3)^{k-1}$

b) $\sum_{k=1}^{\infty} \frac{4}{5}\left(\frac{2}{3}\right)^{k-1}$